Analysis on E, J Model

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1 Introduction

In the paper [3], a $\mathcal{N} = (0, 2)$ SYK-like model, which will be referred to as the J-type model, was discussed. The model describes N chiral super-multiplets and M Fermi super-multiplets with a $(q + 1)$-field interaction. Moreover, higher spin symmetries emerge at specific limits of the parameter $\mu \equiv \frac{M}{N}$.

Another class of models, called the E-type models, which also exhibit the same properties. This paper aims to explore the underlying reasons and reveals the symmetry of E and J superfields at the level of the action.

This paper is organized as follows. In Chapter 2, we review the J-type model discussed in [3]. In Chapter 3, we introduce the calculations of the E-type model and show that both models yield the same characteristic determinant of the kernel matrix, which allows us to follow the analysis in [3] and conclude that the E-type model also has the emergence of higher spin symmetry. In Chapter 4, we argue that the consistency stems from the symmetry at the action level and introduce the $(0, 2)$ Landau-Ginzburg (L-G) model to better show the $E \leftrightarrow J$ symmetry. In Chapter 5, we utilize this symmetry to more efficiently obtain the same characteristic determinant mentioned in Chapter 3. We also show a family of models represented by the parameter s will yield the same results.

2 Review On J-type Model

In this section, we briefly introduce J-type model and its results. Calculation details can be referred in Ch 3.

2.1 J-type Model Setting

We consider the action $S_\Phi^0 + S_\Lambda^0 + S_J$:

$$S_\Phi^0 = -\int dx^2 d\theta^+ d\bar{\theta}^+ \bar{\Phi}\partial_z \Phi \tag{1}$$

$$S_\Lambda^0 = \frac{1}{2}\int dx^2 d\theta^+ d\bar{\theta}^+ \bar{\Lambda}\Lambda \tag{2}$$

$$S_J = \int dx^2 d\theta^+ G(x, \theta^+, \bar{\theta}^+) \tag{3}$$

$$\tag{4}$$

Here, $G(x, \theta^+, \bar{\theta}^+)$ is the Super Potential, composed of the Chiral Super Field Φ, Fermi Super Field Λ, and Gaussian random variables $J_{ia_1\cdots a_q}$:

$$G(x, \theta, \bar{\theta}) \equiv J_{ia_1 \ldots a_q} \Lambda^i \Phi^{a_1} \ldots \Phi^{a_q},$$

Here the summation of repeated index is orderless and Gaussian random variables $J_{ia_1\cdots a_q}$ satisfy:

$$\langle J_{ia_1 \ldots a_q} \rangle = 0$$
$$\langle J_{ia_1 \ldots a_q} J_{ia_1 \ldots a_q} \rangle = \frac{(q-1)!}{N^q} J^2$$

2.2 Field Setup

The component formalism of the Chiral and Fermi Super fields is as follows:

$$\Phi^i = \phi^i + \sqrt{2}\theta^+ \psi_+^i + 2\theta^+ \bar{\theta}^+ \partial_+ \phi^i,$$
$$\bar{\Phi}^i = \bar{\phi}^i - \sqrt{2}\bar{\theta}^+ \bar{\psi}_+^i - 2\theta^+ \bar{\theta}^+ \partial_+ \bar{\phi}^i,$$

$$\Lambda^i = \lambda^i - \sqrt{2}\theta^+ G^i + 2\theta^+\bar{\theta}^+\partial_+\lambda^i - \sqrt{2}\bar{\theta}^+ E^i(\Phi),$$
$$\bar{\Lambda}^i = \bar{\lambda}^i - \sqrt{2}\bar{\theta}^+ \bar{G}^i - 2\theta^+\bar{\theta}^+\partial_+\bar{\lambda}^i - \sqrt{2}\theta^+ \bar{E}^i(\bar{\Phi}).$$

Expanding $E(\Phi)$, where $E^i_{,j} = \partial E^i/\partial\phi^j$:

$$E^i(\Phi) = E^i(\phi) + \sqrt{2}\theta^+ E^i_{,j}\psi^j_+ + 2\theta^+\bar{\theta}^+ E^i_{,j}\partial_+\phi^j,$$

$$\Lambda^i = \lambda^i - \sqrt{2}\theta^+ G^i + 2\theta^+\bar{\theta}^+\partial_+\lambda^i - \sqrt{2}\bar{\theta}^+ E^i(\phi) + 2\theta^+\bar{\theta}^+ E^i_{,j}\psi^j_+.$$
$$\bar{\Lambda}^i = \bar{\lambda}^i - \sqrt{2}\bar{\theta}^+ \bar{G}^i - 2\theta^+\bar{\theta}^+\partial_+\bar{\lambda}^i - \sqrt{2}\theta^+ \bar{E}^i(\bar{\phi}) + 2\theta^+\bar{\theta}^+ \bar{E}^i_{,j}\bar{\psi}^j_+,$$

The J-type model requires $E(\Phi) = 0$, but in E-type model, introduced in Chapter 3, we will consider the case where $E(\Phi) \neq 0$.

2.3 Solve J-type model

We focus on the low-energy ($E \ll J$) conformal region, where the kinetic terms in the action can be neglected, and we pay special attention to the $J_{ia_1\cdots a_q}$ term:

$$S \approx S = \int dx^2 \left(\frac{\sqrt{2}J_{ia_1\ldots a_q}}{(q-1)!}\lambda^i\psi^{a_1}\phi^{a_2}\ldots\phi^{a_q} + \frac{\sqrt{2}J_{ia_1\ldots a_q}}{q!}G^i\phi^{a_1}\ldots\phi^{a_q} + \text{h.c.}\right)$$

Since $J_{ia_1\cdots a_q}$ is a Gaussian random variable, meaningful physical quantities need to consider the ensemble average of the random variable. Introducing the self-energy function Σ and the propagator G in the path integral process, we can derive the correct EOM for this system, namely the Schwinger Dyson Equation (SD eqn):

$$\Sigma^\psi = 2J^2\mu G^\lambda (G^\phi)^{q-1}$$
$$\Sigma^\lambda = 2J^2 (G^\phi)^{q-1} G^\psi$$
$$\Sigma^G = \frac{2J^2}{q}(G^\phi)^q$$
$$\Sigma^\phi = 2J^2\mu((q-1)G^\lambda G^\psi (G^\phi)^{q-2} + (G^\phi)^{q-1}G^G)$$
$$\Sigma^\Psi * G^\Psi = -\mathbb{1} \quad \Psi = \{\phi, \psi, G, \lambda\}$$

2.3.1 Solve SD Eqns

The two-point functions in the region we consider have the general form:

$$G^{\Psi}(z_1, z_2) = \frac{n_\Psi}{(z_1 - z_2)^{2h_\Psi}(\bar{z}_1 - \bar{z}_2)^{2\tilde{h}_\Psi}}$$

The supersymmetric transformation of the propagator are shown as follows, see [2][3][4].

$$\begin{aligned} G^{\psi}(z_1, z_2, \bar{z}_1, \bar{z}_2) &= -2\partial_{z_1} G^{\phi}(z_1, z_2, \bar{z}_1, \bar{z}_2) = 2\partial_{z_2} G^{\phi}(z_1, z_2, \bar{z}_1, \bar{z}_2) \\ G^{G}(z_1, z_2, \bar{z}_1, \bar{z}_2) &= -2\partial_{z_1} G^{\lambda}(z_1, z_2, \bar{z}_1, \bar{z}_2) = 2\partial_{z_2} G^{\lambda}(z_1, z_2, \bar{z}_1, \bar{z}_2) \end{aligned}$$

Substituting the ansatz yields the following relationships for $n, h, \tilde{h}$:

$$n_\psi = -4h_\phi n_\phi, h_\psi = h_\phi + \frac{1}{2}, \bar{h}_\psi = \bar{h}_\phi \tag{5}$$

$$n_G = -4h_\lambda n_\lambda, h_G = h_\lambda + \frac{1}{2}, \bar{h}_G = \bar{h}_\lambda \tag{6}$$

Using the above relationships and the Fourier transform of the SD equation, we can determine $h_\Psi, \tilde{h}_\Psi$ and thus solve the model. The results are as follows:

$$\begin{aligned} h_\phi &= \frac{\mu q - 1}{2\mu q^2 - 2}, \quad h_\psi = \frac{\mu q^2 + \mu q - 2}{2\mu q^2 - 2}, \quad h_\lambda = \frac{q - 1}{2\mu q^2 - 2}, \quad h_G = \frac{\mu q^2 + q - 2}{2\mu q^2 - 2} \\ \tilde{h}_\phi &= \frac{\mu q - 1}{2\mu q^2 - 2}, \quad \tilde{h}_\psi = \frac{\mu q - 1}{2\mu q^2 - 2}, \quad \tilde{h}_\lambda = \frac{\mu q^2 + q - 2}{2\mu q^2 - 2}, \quad \tilde{h}_G = \frac{\mu q^2 + q - 2}{2\mu q^2 - 2} \end{aligned}$$

$$n_\lambda n_\phi^q = -\frac{(q-1)q}{4\pi^2 J^2(\mu q^2 - 1)} \tag{7}$$

2.4 Four Points Function

In the large-N limit, the non-trivial leading-order contribution to the four-point function comes from the sum of ladder diagrams of various lengths.

Using the Kernel K, we can obtain the total contribution of all ladder diagrams:

$$\mathcal{F} = \sum_{n=0}^{\infty} \mathcal{F}_n = \sum_{n=0}^{\infty} K^n \mathcal{F}_0 = \frac{1}{1-K}\mathcal{F}_0.$$

The paper [3] observed the emergence of higher spin symmetry in the Four Point Correlation Function. In this paper, we are primarily interested in the 4-point function $\langle \bar{\phi}^i \phi^j \bar{\phi}^j \phi^j \rangle$, which mixes with $\langle \bar{\phi}^i \phi^j \bar{\psi}^j \psi^j \rangle$, $\langle \bar{\phi}^i \phi^j \bar{\lambda}^j \lambda^j \rangle$, $\langle \bar{\psi}^i \psi^j \bar{\lambda}^j \lambda^j \rangle$, and $\langle \bar{\phi}^i \phi^j \bar{G}^j G^j \rangle$.

From the Feynman diagrams, we know there are the following 9 types of Kernels:

$$\begin{aligned}
K^{\phi\phi}(z_1,z_2,z_3,z_4) &= 2(q-1)J^2\frac{M}{N}G^{\phi}(z_{13})G^{\phi}(z_{24})G^{G}(z_{34})(G^{\phi}(z_{34}))^{q-2} \\
&\quad + 2(q-1)(q-2)J^2\frac{M}{N}G^{\phi}(z_{13})G^{\phi}(z_{24})G^{w}(z_{34})G^{\lambda}(z_{34})(G^{\phi}(z_{34}))^{q-3} \\
K^{\phi\psi}(z_1,z_2,z_3,z_4) &= 2(q-1)J^2\frac{M}{N}G^{\phi}(z_{13})G^{\phi}(z_{24})G^{\lambda}(z_{34})(G^{\phi}(z_{34}))^{q-2} \\
K^{\phi\lambda}(z_1,z_2,z_3,z_4) &= 2(q-1)J^2G^{\phi}(z_{13})G^{\phi}(z_{24})G^{w}(z_{34})(G^{\phi}(z_{34}))^{q-2} \\
K^{\phi G}(z_1,z_2,z_3,z_4) &= 2J^2G^{\phi}(z_{13})G^{\phi}(z_{24})(G^{\phi}(z_{34}))^{q-1} \\
K^{w\phi}(z_1,z_2,z_3,z_4) &= -2(q-1)J^2\frac{M}{N}G^{w}(z_{13})G^{w}(z_{24})G^{\lambda}(z_{34})(G^{\phi}(z_{34}))^{q-2} \\
K^{w\lambda}(z_1,z_2,z_3,z_4) &= -2J^2G^{w}(z_{13})G^{w}(z_{24})(G^{\phi}(z_{34}))^{q-1} \\
K^{\lambda\phi}(z_1,z_2,z_3,z_4) &= -2(q-1)J^2\frac{M}{N}G^{\lambda}(z_{13})G^{\lambda}(z_{24})G^{w}(z_{34})(G^{\phi}(z_{34}))^{q-2} \\
K^{\lambda\psi}(z_1,z_2,z_3,z_4) &= -2J^2\frac{M}{N}G^{\lambda}(z_{13})G^{\lambda}(z_{24})(G^{\phi}(z_{34}))^{q-1} \\
K^{G\phi}(z_1,z_2,z_3,z_4) &= -2J^2\frac{M}{N}G^{G}(z_{13})G^{G}(z_{24})(G^{\phi}(z_{34}))^{q-1}
\end{aligned}$$

By substituting the analytic form of $G^{\Psi}(z,\bar{z})$, we can analyze the form of the kernel's eigenfunctions and solve for the eigenvalues.

$$\begin{aligned}
\Phi^i(z_1,z_2) &= (z_{12})^{h-2h_i}(\bar{z}_{12})^{\tilde{h}-2\tilde{h}_i} \\
K^{(ij)} * \Phi^j &= k^{ij}\Phi^i
\end{aligned}$$

The eigenvalues of the kernel are functions of $q, \mu, h, \tilde{h}$. By combining the kernel eigenvalues into a matrix form, we obtain the kernel matrix as follows:

$$\begin{pmatrix} k^{\phi\phi} & k^{\phi\psi} & k^{\phi\lambda} & k^{\phi G} \\ k^{\psi\phi} & 0 & k^{\psi\lambda} & 0 \\ k^{\lambda\phi} & k^{\lambda\psi} & 0 & 0 \\ k^{G\phi} & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

The characteristic determinant is denoted as $E_c(x, h, \tilde{h}, \mu, q)$. Through this characteristic determinant, we obtain the relationship between $h, \tilde{h}$ and $q, \frac{1}{\mu}$, and observe the emergence of higher spin symmetry. Soon, we are going to see the same characteristic determinant in E-type model.

3 Calculation On E-type Model

This chapter provides a detailed derivation of the E-type model.

$$S = S_\Phi + S_\Lambda, E(\Phi) \neq 0$$

This chapter presents specific calculations to highlight the differences between E, J models. Unlike the J-type model, here $E_a(\Phi_i) = J_{ai_1 i_2,\ldots,i_q}\Phi_{i_1}\ldots\Phi_{i_q}$, and there is no super potential.

3.1 Action

$$\begin{aligned} S_\Phi &\equiv -\int d^2x \int d\theta^+ d\overline{\theta}^+ \overline{\Phi}\partial_{\overline{z}}\Phi \\ &= -\int d^2x \int d\theta^+ d\overline{\theta}^+ \left(-\overline{\phi}2\overline{\theta}^+\theta^+\partial^2\phi + 2\overline{\theta}^+\theta^+\partial\overline{\phi}\partial\phi - 2\overline{\theta}^+\psi\theta^+\partial\psi\right) \\ &= \int d^2x \left(4\overline{\phi}\partial^2\phi - 2\overline{\psi}\partial\psi\right) \end{aligned}$$

$$\begin{aligned}
S_\Lambda &\equiv \frac{1}{2}\int d^2x\, d\theta^+ d\overline{\theta}^+ \overline{\Lambda}\Lambda \\
&= -\frac{1}{2}\int d^2x \times \left(2\overline{\lambda}\partial_z\lambda - 2\overline{G}G - 2\partial_z\overline{\lambda}\lambda + 2\bar{E}(\bar{\phi})E(\phi) + 2\bar{E}^i_{,j}\bar{\psi}^j\lambda_i + 2\bar{\lambda}_i E^i_{,j}\psi^j\right) \\
&= \int d^2x \cdot \left(-2\overline{\lambda}\partial_z\lambda + \overline{G}G - \bar{E}E - \bar{\lambda}_i E^i_{,j}\psi^j - \bar{E}^i_{,j}\bar{\psi}^j\lambda_i\right)
\end{aligned}$$

(The integral results can be verified in [1][4])

In the SYK model, we need to deal with Gaussian integrals concerning $J_{ai_1 i_2,\ldots,i_q}$. However, the appearance of the $\bar{E}E$ term is not convenient for calculations. Therefore, we introduce an auxiliary field B and linearize this term.

$$-\bar{E}E \sim \bar{B}B - BE - \bar{B}\bar{E}$$

Thus, the complete action is

$$S \cong \int d^2x\, \mathcal{L}_{\text{kin}} + \bar{B}B - BE - \bar{B}\bar{E} - \bar{\psi}^j \bar{E}_{a,j}\lambda^a - \bar{\lambda}^a E_{a,j}\psi^j$$

3.2 Complex Gaussian Integral

We need to consider the ensemble average of $J_{\ldots}$, but note that the variables here are complex Gaussian random variables. Before performing the calculations, we need to carefully discuss the properties of complex Gaussian integrals.

1. Properties of complex Gaussian integrals:

   - $J_{\ldots}, \bar{J}_{\ldots}$ are conjugate complex Gaussian variables. When calculating, they are split into real Gaussian variables (x, y), where x, y follow Gaussian distributions with variance σ^2 and the mean μ is 0:

$$J_{\ldots} \to x_{\ldots} + iy_{\ldots} \quad \bar{J}_{\ldots} \to x_{\ldots} - iy_{\ldots}$$

- Following the convention of $s = 2$, $\langle J\bar{J}\rangle = 2\sigma^2$. The weight of the complex Gaussian distribution is assumed to be $\exp(-\frac{x^2+y^2}{2\sigma^2})$.

$$\iint dxdy \exp\left(-\frac{x^2+y^2}{s\sigma^2}\right) = \pi s\sigma^2$$
$$\iint dxdy \exp\left(-\frac{x^2+y^2}{s\sigma^2}(x^2+y^2)\right) = \pi s^2\sigma^4$$
$$\langle J_{...}\overline{J}^{...}\rangle = s\sigma^2$$

- The integration process implicitly involves index pairing. Here is an example for $q = 4$:

  Proof:

$$\sum_{abcd}\int dJ_{abcd}d\bar{J}_{abcd}\exp(-J_{ijkl}X^{ijkl})\exp(-\bar{J}_{i'j'k'l'}\overline{X}^{i'j'k'l'})\times\text{weight}$$
$$=\prod_{abcd}\exp(\cdots)\times\int dJ_{abcd}d\bar{J}_{abcd}\Big(\exp(-J_{abcd}X_{abcd}-\bar{J}_{abcd}\overline{X}_{abcd})\times\text{weights}_{abcd}\Big)$$
$$=\prod_{abcd}\exp(\cdots)\times\exp(\text{Constant}\cdot(X_{abcd}+\overline{X}_{abcd})^2)$$
$$\cong\prod_{abcd}\exp(\cdots)\times\exp\big((X_{abcd}+\overline{X}_{abcd})\cdot(X_{a'b'c'd'}+\overline{X}_{a'b'c'd'})\delta_{aa',bb',cc',dd'}\big)$$

  Therefore, we need to introduce $\prod_{ijkl}\delta_{ia,jb,kc,ld}$ in the subsequent integrals.

- The quantity we need to calculate is

$$\langle\exp(-S)\rangle\sim\langle\exp(-JX-\bar{J}Y)\rangle$$

  Since X, Y are fields that cannot be arbitrarily permuted, it is safest to discuss them in pairs.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\exp(-JX-\bar{J}Y)\,dx\,dy$$

$$\exp\left(-\frac{x^2+y^2}{2\sigma^2}-x(Y+X)-iy(X-Y)\right)$$

  Integrating over x and y separately yields

$$(2\pi\sigma^2)\exp\left(\frac{\sigma^2}{2}(Y+X)^2 - \frac{\sigma^2}{2}(X-Y)^2\right)$$

Simplifying the exponential part:

$$\begin{aligned}
&\frac{\sigma^2}{2}(Y+X)^2 - \frac{\sigma^2}{2}(X-Y)^2 \\
=&\frac{\sigma^2}{2}\left[(Y+X)^2 - (X-Y)^2\right] \\
=&\sigma^2(XY+YX)
\end{aligned}$$

3.3 Field Contraction

After completing the Gaussian integrals, we consider the contraction between identical fields. (The contraction between different types of fields contributes zero) Finally, the dynamics of the system are described by the G^Ψ field. Here, we take the calculation of the Gaussian integral for the $\psi\phi\cdots\phi$ fields as an example, and the case for $B\phi\cdots\phi$ fields is analogous.

3.3.1 Ensemble Average of $\psi\phi\cdots\phi$ Fields

By matching the indices, we obtain

$$\iint d^2z d^2z' \sum_i \bar{\lambda}(\psi_{i_1}\phi_{i_2}\ldots\phi_{i_q} + \ldots + \phi_{i_1}\ldots\phi_{i_{q-1}}\psi_{i_q})|_{z'}\cdot \tag{9}$$

$$(\bar{\psi}_{i_q}\bar{\phi}_{i_{q-1}}\ldots\bar{\phi}_{i_1} + \ldots + \bar{\phi}_{i_q}\bar{\phi}_{i_{q-1}}\ldots\bar{\psi}_{i_1})\lambda|_z \tag{10}$$

$$+ \tag{11}$$

$$\iint d^2z d^2z' \sum_i (\bar{\psi}_{i_q}\bar{\phi}_{i_{q-1}}\ldots\bar{\phi}_{i_1} + \ldots + \bar{\phi}_{i_q}\bar{\phi}_{i_{q-1}}\ldots\bar{\psi}_{i_1})\lambda|_z\cdot \tag{12}$$

$$\bar{\lambda}(\psi_{i_1}\phi_{i_2}\ldots\phi_{i_q} + \ldots + \phi_{i_1}\ldots\phi_{i_{q-1}}\psi_{i_q})|_{z'} \tag{13}$$

1. Rewrite the field bilinears using G through path integrals and perform contractions between fields with the same indices. Specifically, this involves the equivalence relation introduced by the path integral

$$\# \times G(z, z') \sim \sum_a \bar{\Psi}^a(z)\Psi^a(z')$$

Here ($\# = M, N$) represents the number of Chiral Fields and Fermionic Fields respectively.

2. By exchanging the positions of z, z' in the second part and swapping four times fermionic fields, we consider the two parts in eqn 10 - 13 to be identical.

3. The relationship between ordered and unordered summations is as follows:

$$\sum_{1 \leq i_1 \cdots i_q \leq N} = \frac{1}{q!} \sum_{1 \leq i_1 \neq \cdots \neq i_q \leq N} \Rightarrow \frac{1}{q!} \sum_{\{i\}}$$

   In the case of purely fermionic fields, the constraint $\{\neq\}$ can be relaxed. However, for mixed fermionic and bosonic complex fields, it is considered to hold approximately in the large N limit.

4. $G^{\bar{\lambda}} \sim M\lambda\bar{\lambda}$ is equivalent to considering the correlation functions of the $\bar{\lambda}$ field (rather than the λ field), but it is argued in Chapter 3.5.1 that the two are equivalent.

Thus, we consider that the exponential part in $\int \mathcal{D}\Phi\mathcal{D}G\mathcal{D}\Sigma e^{-S}$ contains

$$\iint d^2z d^2z' \frac{MN^q}{(q-1)!} \cdot \langle J_{\dots}\overline{J}_{\dots}\rangle \cdot G^{\bar{\lambda}}(z, z')G^{\psi}(z, z')G^{\phi}(z, z')^{q-1}$$

3.3.2 Ensemble Average of $B\phi \cdots \phi$ Fields

Considering that B, ϕ obey Bose-Einstein statistics, the calculation is simpler than the one above, and we arrive the result as follows:

$$\iint d^2z d^2z' \frac{MN^q}{q!} \cdot \langle J_{\dots}\bar{J}_{\dots}\rangle \cdot G^B(z, z')(G^{\phi}(z, z'))^q$$

3.4 Schwinger-Dyson Equations

Before obtain the equations of motion (EOM) of the system, we need to complete the path integrals of the fields.

3.4.1 Path Integrals

Considering $\langle J_{...}\bar{J}_{...}\rangle = \frac{(q-1)!J^2}{N^q}$, the complete exponent in the path integral is:

$$
\begin{aligned}
&-N\bigg(\Sigma^{\psi}\left(G^{\psi}-\frac{1}{N}\sum\bar{\psi}\psi\right)+\Sigma^{\phi}\left(G^{\phi}-\frac{1}{N}\sum\bar{\phi}\phi\right)\bigg)\\
&-M\bigg(\Sigma^{\bar{\lambda}}\left(G^{\bar{\lambda}}-\frac{1}{M}\sum\lambda\bar{\lambda}\right)+\Sigma^{B}\left(G^{B}-\frac{1}{M}\sum\bar{B}B\right)\bigg)\\
&+\int d^2z\cdot\left(4\overline{\phi}\partial^2\phi-2\overline{\psi}\partial\psi-2\overline{\lambda}\partial\lambda+\overline{G}G+\bar{B}B\right)\\
&+\iint d^2zd^2z'MJ^2G^{\bar{\lambda}}(z,z')G^{\psi}(z,z')G^{\phi}(z,z')^{q-1}\\
&+\iint d^2zd^2z'\frac{J^2M}{q}G^{B}(z,z')(G^{\phi}(z,z'))^{q}
\end{aligned}
$$

First, consider the scalar case. Our equations will encounter Gaussian integrals of complex scalar fields. Using the formula

$$\prod\iint\mathcal{D}\bar{\phi}_i\,\mathcal{D}\phi_i\,e^{-\bar{\phi}_i\,h_{ij}\,\phi_j}=\frac{1}{\det(h)}$$

(for spinor fields, the integral result corresponds to $h\Rightarrow h^{-1}$)

We'll get following structure, # is the operator obtained by matching the bilinear form.

$$
\begin{aligned}
&\iint\mathcal{D}\phi^{\dagger}\mathcal{D}\phi\exp\left(-\sum\phi^{\dagger}(\#-\Sigma)\phi-N\Sigma G+\cdots\right)\\
=&\exp\left(-N\ln(\det(\#-\Sigma))-N\Sigma G+\cdots\right)
\end{aligned}
$$

3.4.2 First Set of SD Equations:

Variation with respect to Σ, we'll have

$$G=\frac{1}{\#-\Sigma}$$

We focus on the low energy region $w \ll 1 \ll J$, so we can drop the # term.

Note that, since the first SD equation for the G field is $G^G = \frac{1}{-2-\Sigma^G}$, but due to the lack of a G field coupled to $J_{...}$, we can infer from the second SD equation that on-shell $\Sigma^G = 0$, so it does not contain any dynamics.

3.4.3 Second Set of SD Equations

Variation with respect to G, we'll have

$$
\begin{aligned}
\Sigma^{\psi} &= \mu J^2 G^{\overline{\lambda}} (G^{\phi})^{q-1} \\
\Sigma^{\phi} &= J^2 \mu \left((q-1) G^{\overline{\lambda}} G^{\psi} (G^{\phi})^{q-2} + G^B \left(G^{\phi}\right)^{q-1} \right) \\
\Sigma^{\overline{\lambda}} &= J^2 G^{\psi} (G^{\phi})^{q-1} \\
\Sigma^B &= \frac{J^2}{q} (G^{\phi})^q
\end{aligned}
$$

where $\frac{M}{N} \equiv \mu$.

3.5 Solve SD Eqn

Obtaining the Schwinger-Dyson (SD) equations in principle allows us to solve the system.

The homogeneity of the SD equations suggests the use of a conformal ansatz for G.

$$
G^i(z_1, z_2) = \frac{n_i}{(z_1 - z_2)^{2h_i} (\bar{z}_1 - \bar{z}_2)^{2\tilde{h}_i}}
$$

Here, we present the detailed calculations of each G^{Ψ} and explore its properties.

3.5.1 Relation between $G^{\bar{\Psi}}$ 与 G^{Ψ}

We are going to show

$$G^{\bar{\Psi}}(z) = G^{\Psi}(z) = (-1)^{2s} G^{\Psi}(-z)$$
$$h_{\bar{\Psi}} = h_{\Psi}$$
$$\tilde{h}_{\bar{\Psi}} = \tilde{h}_{\Psi}$$
$$n_{\bar{\Psi}} = n_{\Psi}$$

Proof

$$\begin{aligned} G^{\bar{\Psi}}(z_1, z_2) \equiv& \langle \Psi(z_1) \bar{\Psi}(z_2) \rangle \\ =& (-1)^{2s} \langle \bar{\Psi}(z_2) \Psi(z_1) \rangle \\ =& (-1)^{2(h_\Psi - \tilde{h}_\Psi)} G^{\Psi}(z_2, z_1) \\ \therefore G^{\bar{\Psi}}(z) =& (-1)^{2(h_\Psi - \tilde{h}_\Psi)} G^{\Psi}(-z) \end{aligned}$$

The second step uses the statistical properties of particles, while the last step utilizes the form of the Ansatz.

$$\because -1 = e^{i\pi}, \overline{-1} = e^{-i\pi}$$
$$\therefore -1^x \times \overline{(-1)}^x = 1$$
$$\therefore (-z)^{2h_\Psi} (\overline{-z})^{2\tilde{h}_\Psi} = (-1)^{2h_\Psi - 2\tilde{h}_\Psi} (z)^{2h_\Psi} (\bar{z})^{2\tilde{h}_\Psi}$$

Using the properties discussed above we can verified that $G^{\bar{\Psi}}(z) = G^{\Psi}(z) = (-1)^{2s} G^{\Psi}(-z)$

$$\frac{n_{\bar{\Psi}}}{z^{2h_{\bar{\Psi}}} \bar{z}^{2\tilde{h}_{\bar{\Psi}}}} = (-1)^{2(h_\Psi - \tilde{h}_\Psi)} \frac{n_\Psi}{(-z)^{2h_\Psi} (\overline{-z})^{2\tilde{h}_\Psi}} = \frac{n_\Psi}{z^{2h_\Psi} z^{2\tilde{h}_\Psi}}$$

Thus, we can replace $G^{\bar{\lambda}}(z)$ with $G^{\lambda}(z)$ in the SD equations. Since the $G^{\bar{\lambda}}$ field is no longer involved, we can replace $\Sigma^{\bar{\lambda}}$ with Σ^{λ}. The simplified SD equations are as follows:

$$\Sigma^\psi = \mu J^2 G^\lambda (G^\phi)^{q-1} \tag{14}$$

$$\Sigma^\phi = J^2 \mu \left((q-1) G^\lambda G^\psi (G^\phi)^{q-2} + G^B \left(G^\phi\right)^{q-1} \right) \tag{15}$$

$$\Sigma^\lambda = J^2 G^\psi (G^\phi)^{q-1} \tag{16}$$

$$\Sigma^B = \frac{J^2}{q} (G^\phi)^q \tag{17}$$

3.5.2 Fourier Transform

The Fourier space provides a simpler way to solve the SD equations

$$\Sigma(w) G(w) = -1$$

According to the second set of SD equations, Σ can be written in the form of $\frac{n_\Sigma}{z^{2h_\Sigma} \bar{z}^{2\tilde{h}_\Sigma}}$ (or its combinations), where the conformal weights are $h_\Sigma, \tilde{h}_\Sigma$.

Using the following Fourier transform:

$$\frac{1}{z^{2h} \bar{z}^{2\bar{h}}} \Rightarrow \frac{\pi}{i^{2(h-\bar{h})} 2^{2\bar{h}+2h-2}} \cdot \frac{\Gamma(1-2h)}{\Gamma(2\bar{h})} \frac{1}{p^{(2\bar{h}+1)} \bar{p}^{2h+1}}$$

we can proceed to solve the equations.

$$\mathcal{F}\left[\frac{n_\Sigma}{z^{2h_\Sigma} \bar{z}^{2\tilde{h}_\Sigma}}\right] \mathcal{F}\left[\frac{n_G}{z^{2h_G} \bar{z}^{2\tilde{h}_G}}\right] = -1$$

$$\mathcal{F}\left[\frac{n_\Sigma}{z^{2(1-h_G)} \bar{z}^{2(1-\tilde{h}_G)}}\right] \mathcal{F}\left[\frac{n_G}{z^{2h_G} \bar{z}^{2\tilde{h}_G}}\right] = -1$$

$$(-1)^{2h_G - 2\tilde{h}_G + 1} \frac{n_\Sigma n_G \pi^2}{(2h_G - 1)(2\tilde{h}_G - 1)} = -1$$

1. The first step uses the matching of the exponents of p to obtain the relations

$$h_\Sigma + h_G = 1 \quad \tilde{h}_\Sigma + \tilde{h}_G = 1$$

2. The second step uses the statistical properties of the fields, $2(h - \bar{h}) = 2s \in \mathbb{Z}$ and arrives

$$-\frac{\pi^2}{(2\bar{h}-1)(2h-1)} \cdot \frac{\sin(2\pi\bar{h})}{\sin(2\pi\bar{h}+2\pi s)}$$

3.5.3 Exponent Matching for the First Set of SD Equations:

From the above equations, we can derive two independent sets of conformal weight relations from the second set of SD equations:

$$\begin{aligned} h_\psi + h_\lambda + (q-1)h_\phi &= 1 \\ \tilde{h}_\psi + \tilde{h}_\lambda + (q-1)\tilde{h}_\phi &= 1 \\ h_B + qh_\phi &= 1 \\ \tilde{h}_B + q\tilde{h}_\phi &= 1 \end{aligned}$$

Combining the four equations in eqn 2.3.1, we can express all $h, \tilde{h}$ in terms of $h_\phi, \tilde{h}_\phi$. And the Fourier transforms of equations 16 and 17 allow us to solve for $h_\phi, \tilde{h}_\phi$:

$$\pi^2 J^2 n_\psi n_\lambda n_\phi^{q-1} \cdot (-1)^{2h_\psi - 2\tilde{h}_\psi + 1} \cdot \frac{\mu}{(2h_\psi - 1)(2\tilde{h}_\psi - 1)} = -1 \tag{18}$$

$$\pi^2 J^2 n_\psi n_\lambda n_\phi^{q-1} \cdot (-1)^{2h_\lambda - 2\tilde{h}_\lambda + 1} \cdot \frac{1}{(2h_\lambda - 1)(2\tilde{h}_\lambda - 1)} = -1 \tag{19}$$

$$\frac{\mu}{2h_\phi(2\tilde{h}_\phi - 1)} = \frac{1}{(1-2\tilde{h}_\phi q)(-2h_\phi q)} \tag{20}$$

Since ϕ is a scalar field, we have $h_\phi = \tilde{h}_\phi$, and thus we can solve for all $h, \tilde{h}$:

$$\begin{gathered} \tilde{h}_\phi = h_\phi = \frac{\mu q - 1}{2\mu q^2 - 2} \\ \tilde{h}_\psi = \frac{\mu q - 1}{2\mu q^2 - 2} \quad h_\psi = \frac{\mu q^2 + \mu q - 2}{2\mu q^2 - 2} \\ \tilde{h}_B = h_B = \frac{\mu q^2 + q - 2}{2\mu q^2 - 2} \\ \tilde{h}_\lambda = \frac{\mu q^2 + q - 2}{2\mu q^2 - 2} \quad h_\lambda = \frac{q-1}{2\mu q^2 - 2} \end{gathered}$$

3.5.4 Obtaining the n Relations

Since we know that $n_\Sigma n_G = \frac{(-1)^{-2s}(2h_G-1)(2\tilde{h}_G-1)}{\pi^2}$, it leads to the following relations:

$$n_\lambda n_\phi^q = -\frac{(q-1)q}{2\pi^2 J^2 \left(\mu q^2 - 1\right)}$$

$$n_B n_\phi^q = \frac{(q-1)^2 q}{\pi^2 J^2 (\mu q^2 - 1)^2}$$

There are two sets of expressions for n_Ψ is because B is an auxiliary field without super symmetric transformations. However, it is worth noting the similarity between these expressions and the results of the J-type model in ch 7.

3.6 Four Points Function

By calculating with Feynman diagrams, we obtain the following expression for the kernel:

$$\begin{aligned}
K^{\phi\phi}(z_1,z_2,z_3,z_4) &= (q-1)J^2\frac{M}{N}G^\phi(z_{13})G^\phi(z_{24})G^B(z_{34})(G^\phi(z_{34}))^{q-2}\\
&+(q-1)(q-2)J^2\frac{M}{N}G^\phi(z_{13})G^\phi(z_{24})G^\psi(z_{34})G^\lambda(z_{34})(G^\phi(z_{34}))^{q-3}\\
K^{\phi\psi}(z_1,z_2,z_3,z_4) &= (q-1)J^2\frac{M}{N}G^\phi(z_{13})G^\phi(z_{24})G^\lambda(z_{34})(G^\phi(z_{34}))^{q-2}\\
K^{\phi\lambda}(z_1,z_2,z_3,z_4) &= (q-1)J^2G^\phi(z_{13})G^\phi(z_{24})G^\psi(z_{34})(G^\phi(z_{34}))^{q-2}\\
K^{\phi B}(z_1,z_2,z_3,z_4) &= J^2G^\phi(z_{13})G^\phi(z_{24})(G^\phi(z_{34}))^{q-1}\\
K^{\psi\phi}(z_1,z_2,z_3,z_4) &= -(q-1)J^2\frac{M}{N}G^\psi(z_{13})G^\psi(z_{24})G^\lambda(z_{34})(G^\phi(z_{34}))^{q-2}\\
K^{\psi\lambda}(z_1,z_2,z_3,z_4) &= -J^2G^\psi(z_{13})G^\psi(z_{24})(G^\phi(z_{34}))^{q-1}\\
K^{\lambda\phi}(z_1,z_2,z_3,z_4) &= -(q-1)J^2\frac{M}{N}G^\lambda(z_{13})G^\lambda(z_{24})G^\psi(z_{34})(G^\phi(z_{34}))^{q-2}\\
K^{\lambda\psi}(z_1,z_2,z_3,z_4) &= -J^2\frac{M}{N}G^\lambda(z_{13})G^\lambda(z_{24})(G^\phi(z_{34}))^{q-1}\\
K^{B\phi}(z_1,z_2,z_3,z_4) &= -J^2\frac{M}{N}G^B(z_{13})G^B(z_{24})(G^\phi(z_{34}))^{q-1}
\end{aligned}$$

1. To ensure that the four-point function is connected in the correct order, one should in principle carefully handle G^{Ψ}, $G^{\bar{\Psi}}$, and z_{ij} or z_{ji}. However, using our previous discussion in sec 3.5.1, it is only necessary to carefully handle the signs.

2. To ensure that the fields and their conjugates are connected in the correct order, we need to cross the up and down rails. This will carry a negative sign for fermionic external legs.

3.7 Eigenvalues of the Kernel

The form of the eigenfunctions and the eigenvalues are given by

$$\begin{aligned}&\Phi^i(z_1,z_2) = (z_{12})^{h-2h_i}(\bar{z}_{12})^{\tilde{h}-2\tilde{h}_i}, \quad i=\phi,\psi,\lambda,G\\ &K^{(ij)} * \Phi^j = k^{ij}\Phi^i.\end{aligned}$$

The integrand for the eigenvalue calculation is:

$$\iint d^2z_3 d^2z_4 \ K^{(ij)}(z_1,z_2,z_3,z_4)\Phi^j(z_3,z_4) = k^{ij}\Phi^i(z_1,z_2)$$

$$K^{ij} * \Phi^j \propto z_{13}^{-2h_i}\bar{z}_{13}^{-2\tilde{h}_i} \cdot z_{24}^{-2h_i}\bar{z}_{24}^{-2\tilde{h}_i} \cdot z_{34}^{h-2h_j-2h^*}\bar{z}_{34}^{\tilde{h}-2\tilde{h}_j-2\tilde{h}^*}$$

(where h^* $\tilde{h}^*$ represent the accumulated conformal weights of the Green functions exchanged between the two rails)

Using following formula, we can handle the integral

$$\begin{aligned}&\int d^2y(y-t_0)^{a+n}(\bar{y}-\bar{t}_0)^a(t_1-y)^{b+m}(\bar{t}_1-\bar{y})^b\\ &\quad =(t_0-t_1)^{a+n+b+m+1}(\bar{t}_0-\bar{t}_1)^{a+b+1} \quad \times\\ &\quad\quad \pi\frac{\Gamma(a+1)\Gamma(b+1)\Gamma(-a-b-m-n-1)}{\Gamma(a+b+2)\Gamma(-a-n)\Gamma(-b-m)}\end{aligned}$$

3.7.1 Integral Processing

Step 1: Integrate over $z_3, \bar{z}_3$ equals to directly substitute

$$\begin{aligned}
&t_0 \Rightarrow z_4, && t_1 \Rightarrow z_1,\\
&m \Rightarrow -2(h_i - \tilde{h}_i), && b \Rightarrow -2\tilde{h}_i,\\
&a \Rightarrow \tilde{h} - 2\tilde{h}_j - 2\tilde{h}^*, && n \Rightarrow (h - \tilde{h}) - 2(h^* - \tilde{h}^*) - 2(h_j - \tilde{h}_j).
\end{aligned}$$

The resulting function form is:

$$z_{41}^{h-2h_j-2h_i-2h^*+1}\bar{z}_{41}^{\tilde{h}-2\tilde{h}_j-\tilde{h}_i-2\tilde{h}^*+1} \cdot z_{24}^{-2h_i}\bar{z}_{24}^{-2\tilde{h}_i}.$$

Step 2: Integrate over $z_4, \bar{z}_4$equals to directly substitute

$$\begin{aligned}
&t_0 \Rightarrow z_1, && t_1 \Rightarrow z_2,\\
&b \Rightarrow -2\tilde{h}_i, && m \Rightarrow -2(h_i - \tilde{h}_i),\\
&a \Rightarrow \tilde{h} - 2\tilde{h}_j - 2\tilde{h}_i - 2\tilde{h}^* + 1, && n \Rightarrow (h - \tilde{h}) - 2(h^* - \tilde{h}^*) - 2(h_j - \tilde{h}_j) - 2(h_i - \tilde{h}_i).
\end{aligned}$$

The resulting function form after integration is

$$z_{12}^{h-2h_i+(2-2h_j-2h_i-2h^*)}\bar{z}_{12}^{\tilde{h}-2\tilde{h}_i+(2-2\tilde{h}_j-2\tilde{h}_i-2\tilde{h}^*)}.$$

Another interpretation of the p balance relation, introduced in CH3.5.3, is to obtain the conformal weight normalization for all Green functions connected to each vertex.

$$2 - 2h_j - 2h_i - 2h^* = 0$$

Thus, the final function form we obtain is $z_{12}^{h-2h_i}\bar{z}_{12}^{\tilde{h}-2\tilde{h}_i}$.

Since the form of the eigenfunctions is correct, the process of calculating the eigenvalues, only needs to calculate the products of two sets of the following functions:

$$F[a, b, m, n] = \pi\frac{\Gamma(a+1)\Gamma(b+1)\Gamma(-a-b-m-n-1)}{\Gamma(a+b+2)\Gamma(-a-n)\Gamma(-b-m)}$$

Denote the integrand of the complex convolution as $K^{ij}\Phi = z_{13}^{X}z_{24}^{Y}z_{34}^{Z} \cdot \bar{z}_{13}^{\tilde{X}}\bar{z}_{24}^{\tilde{Y}}\bar{z}_{34}^{\tilde{Z}}$, the final result of eigenvalues are

$$F[\tilde{Z}, \tilde{X}, X - \tilde{X}, Z - \tilde{Z}] \cdot F[\tilde{X} + \tilde{Z} + 1, \tilde{Y}, Y - \tilde{Y}, Z + X - \tilde{Z} - \tilde{X}]$$

3.7.2 Eigenvalue Calculation Results

$$k^{\phi\phi} = \frac{\mu(q-1)^2 q(\mu q^2 - 2\mu q + 1)\Gamma\left(\frac{(q-1)q\mu}{q^2\mu-1}\right)^2 \Gamma\left(-\frac{h\mu q^2+\mu q+h-1}{q^2\mu-1}\right)\Gamma\left(\tilde{h} - \frac{(q-1)q\mu}{q^2\mu-1}\right)}{4(\mu q^2-1)^2\Gamma\left(\frac{q\mu-1}{q^2\mu-1}\right)^2 \Gamma\left(\frac{h\mu q^2-2\mu q^2+\mu q-h+1}{1-q^2\mu}\right)\Gamma\left(\tilde{h}+\frac{(q-1)q\mu}{q^2\mu-1}\right)}$$

$$k^{\phi\psi} = -\frac{\mu(q-1)^2 q\Gamma\left(\frac{(q-1)q\mu}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+\mu q+h-1}{q^2\mu-1}\right)\Gamma\left(\tilde{h} - \frac{(q-1)q\mu}{q^2\mu-1}\right)}{8(\mu q^2-1)\Gamma\left(\frac{q\mu-1}{q^2\mu-1}\right)^2 \Gamma\left(\frac{h\mu q^2-2\mu q^2+\mu q-h+1}{1-q^2\mu}\right)\Gamma\left(\tilde{h}+\frac{(q-1)q\mu}{q^2\mu-1}\right)}$$

$$k^{\phi\lambda} = -\frac{2\pi^2 J^2 (q-1) n_\phi^{q+1}(\mu q-1)\Gamma\left(\frac{(q-1)q\mu}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+\mu q+h-1}{q^2\mu-1}\right)\Gamma\left(\tilde{h} - \frac{(q-1)q\mu}{q^2\mu-1}\right)}{(\mu q^2-1)\Gamma\left(\frac{q\mu-1}{q^2\mu-1}\right)^2 \Gamma\left(\frac{h\mu q^2-2\mu q^2+\mu q-h+1}{1-q^2\mu}\right)\Gamma\left(\tilde{h}+\frac{(q-1)q\mu}{q^2\mu-1}\right)}$$

$$k^{\phi B} = -\frac{\pi^2 J^2 n_\phi^{q+1}\Gamma\left(\frac{(q-1)q\mu}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+\mu q+h-1}{q^2\mu-1}\right)\Gamma\left(\tilde{h} - \frac{(q-1)q\mu}{q^2\mu-1}\right)}{\Gamma\left(\frac{q\mu-1}{q^2\mu-1}\right)^2 \Gamma\left(\frac{h\mu q^2-2\mu q^2+\mu q-h+1}{1-q^2\mu}\right)\Gamma\left(\tilde{h}+\frac{(q-1)q\mu}{q^2\mu-1}\right)}$$

$$k^{\psi\phi} = \frac{\mu(q-1)^2 q(\mu q-1)^2\Gamma\left(\frac{(q-1)q\mu}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+\mu q^2+\mu q+h-2}{q^2\mu-1}\right)\Gamma\left(\tilde{h} - \frac{(q-1)q\mu}{q^2\mu-1}\right)}{2(\mu q^2-1)^3\Gamma\left(\frac{\mu q^2+\mu q-2}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+(q-1)\mu q+h}{q^2\mu-1}\right)\Gamma\left(\tilde{h}+\frac{(q-1)q\mu}{q^2\mu-1}\right)}$$

$$k^{\psi\lambda} = -\frac{4\pi^2 J^2 n_\phi^{q+1}(\mu q-1)^2\Gamma\left(\frac{(q-1)q\mu}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+\mu q^2+\mu q+h-2}{q^2\mu-1}\right)\Gamma\left(\tilde{h} - \frac{(q-1)q\mu}{q^2\mu-1}\right)}{(\mu q^2-1)^2\Gamma\left(\frac{\mu q^2+\mu q-2}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+(q-1)\mu q+h}{q^2\mu-1}\right)\Gamma\left(\tilde{h}+\frac{(q-1)q\mu}{q^2\mu-1}\right)}$$

$$k^{\lambda\phi} = \frac{\mu(q-1)^3 q^2 n_\phi^{-q-1}(\mu q-1)\Gamma\left(\frac{1-q}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+q+h-1}{q^2\mu-1}\right)\Gamma\left(\frac{q+\tilde{h}(q^2\mu-1)-1}{q^2\mu-1}\right)}{32\pi^2 J^2(\mu q^2-1)^3\Gamma\left(\frac{q-1}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+2\mu q^2-q+h-1}{q^2\mu-1}\right)\Gamma\left(\frac{-q+\tilde{h}(q^2\mu-1)+1}{q^2\mu-1}\right)}$$

$$k^{\lambda\psi} = -\frac{\mu(q-1)^2 q^2 n_\phi^{-q-1}\Gamma\left(\frac{1-q}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+q+h-1}{q^2\mu-1}\right)\Gamma\left(\frac{q+\tilde{h}(q^2\mu-1)-1}{q^2\mu-1}\right)}{64\pi^2 J^2(\mu q^2-1)^2\Gamma\left(\frac{q-1}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+2\mu q^2-q+h-1}{q^2\mu-1}\right)\Gamma\left(\frac{-q+\tilde{h}(q^2\mu-1)+1}{q^2\mu-1}\right)}$$

$$k^{B\phi} = -\frac{\mu(q-1)^4 q^2 n_\phi^{-q-1}\Gamma\left(\frac{1-q}{q^2\mu-1}\right)^2 \Gamma\left(\frac{-h\mu q^2+\mu q^2+q+h-2}{q^2\mu-1}\right)\Gamma\left(\frac{q+\tilde{h}(q^2\mu-1)-1}{q^2\mu-1}\right)}{16\pi^2 J^2(\mu q^2-1)^4\Gamma\left(\frac{\mu q^2+q-2}{q^2\mu-1}\right)^2\Gamma\left(\frac{-h\mu q^2+\mu q^2-q+h}{q^2\mu-1}\right)\Gamma\left(\frac{-q+\tilde{h}(q^2\mu-1)+1}{q^2\mu-1}\right)}$$

Its greatly similar to J-type model's result and the kernel matrix for E-type model is as follows. The representation is on $\{\phi, \lambda, \psi, B\}$ but uses the result in [3] denoted as $\tilde{k}$.

$$\begin{pmatrix} \tilde{k}^{\phi\phi} & \tilde{k}^{\phi\psi} & \frac{\tilde{k}^{\phi\lambda}}{2} & \frac{\tilde{k}^{\phi G}}{2} \\ \tilde{k}^{\psi\phi} & 0 & \frac{\tilde{k}^{\psi\lambda}}{2} & 0 \\ 2\tilde{k}^{\lambda\phi} & 2\tilde{k}^{\lambda\psi} & 0 & 0 \\ 2\tilde{k}^{G\phi} & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

We can see the characteristic determinant is exactly same as the one from matrix in eqn 8

4 Duality of the $(0,2)$ Landau-Ginzburg Model

The following discussion is primarily based on this paper[2], discussing the symmetry of $E \leftrightarrow J$ on the $\mathcal{L}$ level.

4.1 Action

The complete $(0,2)$ L-G action includes $S^0_\Lambda + S^0_\Phi + S_J$ with $E \neq 0$

$$S_{\text{kin}} = S_\Phi + S_\Lambda = -\int d^2x\, d\theta^+ d\overline{\theta}^+ \overline{\Phi}\partial_{\bar{z}}\Phi + \frac{1}{2}\int d^2x\, d\theta^+ d\overline{\theta}^+ \overline{\Lambda}\Lambda \tag{22}$$

$$S_J = -\int d^2y d\theta^+ \Lambda^i J_i(\Phi)|_{\overline{\theta}^+=0} + \text{h.c.} \tag{23}$$

As we already seen the action for S_Φ and S_Λ in CH 3, we'll analysis S_J in following.

4.1.1 S_J Integral:

Require $\bar{\theta}^+ = 0$. For the convenience of calculation, we can expand Φ, $J(\Phi)$, and Λ:

$$\Phi^i|_{\bar\theta^+=0} = \phi^i + \sqrt{2}\theta^+\psi_+^i,$$
$$J_i(\Phi)|_{\bar\theta^+=0} = J_i(\phi) + \sqrt{2}\theta^+ J_{i,j}\psi_+^j, \quad \text{where} \quad J_{i,j} = \frac{\partial J_i}{\partial \Phi^j},$$
$$\Lambda^i|_{\bar\theta^+=0} = \lambda^i - \sqrt{2}\theta^+ G^i.$$

We can obtain the contribution from the θ^+ part:

$$\Lambda^i J_i|_{\bar\theta^+=0} = \lambda^i J_i + \sqrt{2}\theta^+(-\lambda^i J_{i,j}\psi_+^j - G^i J_i)$$

Finally, we get:

$$S_J = \sqrt{2}\int d^2x \cdot (\lambda^i J_{i,j}\psi_+^j + G^i J_i)$$

Derivation of the conjugate part:

$$S_J^{\text{hc}} = \int d^2x\, d\bar\theta^+ \bar J(\bar\Phi)\bar\Lambda$$
$$\bar\Phi^i|_{\theta^+=0} = \bar\phi^i - \sqrt{2}\bar\theta^+\bar\psi_+^i,$$
$$\bar J_i(\bar\Phi)|_{\theta^+=0} = \bar J_i(\bar\phi) - \sqrt{2}\bar\theta^+\bar J_{i,j}\bar\psi_+^j, \quad \text{where} \quad \bar J_{i,j} = \frac{\partial \bar J_i}{\partial \bar\phi^j},$$
$$\bar\Lambda^i|_{\theta^+=0} = \bar\lambda^i - \sqrt{2}\bar\theta^+\bar G^i.$$

Thus, we can obtain:

$$S_J^{\text{hc}} = \sqrt{2}\int d^2x \left(\bar G^i \bar J_i + \bar J_{i,j}\bar\psi_+^j\bar\lambda^i\right)$$

$$S_J = \sqrt{2}\int d^2x \cdot (\lambda^i J_{i,j}\psi_+^j + G^i J_i + \bar\psi_+^j \bar J_{i,j}\bar\lambda^i + \bar J_i \bar G^i)$$

Complete $\mathcal{L}$

$$\left(4\overline{\phi}\partial^2\phi - 2\overline{\psi}\partial\psi\right) \tag{24}$$
$$+\left(-2\overline{\lambda}\partial_z\lambda + \overline{G}G - \bar E E - \bar\lambda_i E^i_{,j}\psi^j - \bar E^i_{,j}\bar\psi^j\lambda_i\right) \tag{25}$$
$$+\sqrt{2}\left(\lambda^i J_{i,j}\psi_+^j + G^i J_i + \bar\psi_+^j \bar J_{i,j}\bar\lambda^i + \bar J_i\bar G^i\right) \tag{26}$$

4.2 $E \leftrightarrow J$ Symmetry

Remark:

We can see that the G field has no kinetic terms and is similar to an auxiliary field. The equations of motion for G and $\bar{G}$ yield:

$$
\begin{aligned}
G^i &= -\sqrt{2}\bar{J}_i, \\
\bar{G}^i &= -\sqrt{2}J_i.
\end{aligned}
$$

Thus, we can obtain the contributions from the G-J coupling terms and the G field itself: $+2\bar{J}J$.

$\mathcal{L}$ without the G field

$$
\begin{aligned}
\mathcal{L} = &\ \mathrm{kin}_{\phi,\psi,\lambda} + \\
&- 2\bar{J}J - \bar{\psi}_+^j(-\sqrt{2}\bar{J}^i_{,j})\bar{\lambda}_i - \lambda_i(-\sqrt{2}J^i_{,j})\psi_+^j \\
&- \bar{E}E - \bar{\psi}^j\bar{E}^i_{,j}\lambda_i - \bar{\lambda}_i E^i_{,j}\psi^j
\end{aligned}
$$

We can see that there is a symmetry transformation:

$$
\boxed{E \Leftrightarrow -\sqrt{2}J, \quad \lambda \leftrightarrow \bar{\lambda}}
$$

In addition, there are constraints on E and J field. [2][4][1]:

$$
E \cdot J = 0
$$

This is naturally satisfied for the subsequent discussions of $E = 0, J \neq 0$ and $E \neq 0, J = 0$.

5 J-type and E-type SYK Models

By substituting the specific E, J-type model super field, and introducing auxiliary fields in the E-type model, we can intuitively see the symmetry condition for E, J models at the level of the action. We can generalize this condition when we focus on the discussion on characteristic determinant of kernel matrix.

Compare the action obtained in CH2,3 for J, E-type models, we can see

$$S^E \int d^2x \mathcal{L}_{\text{kin}} + \bar{B}B - BE - \bar{B}\bar{E} - \bar{\psi}^j \bar{E}_{a,j}\lambda^a - \bar{\lambda}^a E_{a,j}\psi^j \tag{27}$$

$$S^J = \int d^2x \mathcal{L}_{\text{kin}}(\supset +\bar{G}G) + \sqrt{2}GJ + \sqrt{2}\bar{G}\bar{J} + \sqrt{2}\bar{\psi}^j \bar{J}_{a,j}\bar{\lambda}^a + \sqrt{2}\lambda^a J_{a,j}\psi^j \tag{28}$$

The fields in the E, J-type super potential models have the following correspondence at the level of the action

$$G_J \Leftrightarrow B_E \tag{29}$$

$$\lambda_J \Leftrightarrow \bar{\lambda}_E \tag{30}$$

$$-\sqrt{2}J \Leftrightarrow E \tag{31}$$

However, in our case, $E(\Phi) = J(\Phi)$, which do not fully satisfy the duality conditions discussed previously. The overall sign of $-\sqrt{2}$ does affect the value of kenels. However, it would not affect the characteristic determinant of the kernel matrix.

This chapter demonstrates that the differences lie only in the coefficients, without presenting the specific and complex calculations. At the end of this chapter, a simplified method will be introduced to show how to quickly obtain the differences in coefficients.

5.1 Comparison of SD Equations

We can calculate the SD eqn as follows. The blue ones are the SD equations of the E-type model, and the black ones are the SD equations of the J-type model.

$$\Sigma^{\psi} = \mu J^2 G^{\bar{\lambda}} (G^{\phi})^{q-1} \qquad \Sigma^{\psi} = 2J^2 \mu G^{\lambda} (G^{\phi})^{q-1}$$
$$\Sigma^{\lambda} = J^2 G^{\psi} (G^{\phi})^{q-1} \qquad \Sigma^{\lambda} = 2J^2 (G^{\phi})^{q-1} G^{\psi}$$
$$\Sigma^{B} = \frac{J^2}{q} (G^{\phi})^{q} \qquad \Sigma^{G} = \frac{2J^2}{q} (G^{\phi})^{q}$$

$$\Sigma^{\phi} = J^2 \mu ((q-1) G^{\bar{\lambda}} G^{\psi} (G^{\phi})^{q-2} + (G^{\phi})^{q-1} G^{B})$$
$$\Sigma^{\phi} = 2J^2 \mu ((q-1) G^{\lambda} G^{\psi} (G^{\phi})^{q-2} + (G^{\phi})^{q-1} G^{G})$$

5.2 Comparison of $h_\Psi, \tilde{h}_\Psi, n_\Psi$

Since h and $\tilde{h}$ are only related to the conformal weight of fields in the Fourier transform process, and are independent of the coefficients in the SD equations.

As for n_Ψ:

$$n_\lambda n_\phi^q = -\frac{(q-1)q}{4\pi^2 J^2 (\mu q^2 - 1)}$$
$$\Downarrow$$
$$n_\lambda n_\phi^q = -\frac{(q-1)q}{2\pi^2 J^2 (\mu q^2 - 1)}$$
$$n_B n_\phi^q = \frac{(q-1)^2 q}{\pi^2 J^2 (\mu q^2 - 1)^2} \tag{32}$$

Differences:

1. In eqn 32, it can be seen that the n factors differ by a factor of two.
2. Since there is no supersymmetric transformation for the B field, the relationship between the B and λ fields cannot be constructed. However, the B field is greatly similar to the G field shown in J-type model.

$$n_B = -4 \cdot \frac{q-1}{2(\mu q^2 - 1)} n_\lambda = -4 h_\lambda n_\lambda$$
$$n_G = -4 h_\lambda n_\lambda$$
$$n_B = n_G$$

5.3 Comparison of $S(J_{ai_1\cdots i_q})$

Comparison of the interaction fields coupled with $J_{ai_1\cdots i_q}$:

$$-J_{ai_1\cdots i_q}\cdot\left(q\bar{\lambda}\psi_{i_1}\phi_{i_2}\cdots\phi_{i_q}+B^a\phi_{i_1}\phi_{i_2}\cdots\phi_{i_q}\right)$$

$$\sqrt{2}J_{ia_1\cdots a_q}\cdot\left(q\lambda^i\psi^{a_1}\phi^{a_2}\cdots\phi^{a_q}+G^i\phi^{a_1}\cdots\phi^{a_q}\right)$$

Differences:

1. Although $\bar{\lambda}$ field is different from λ, in the Ansatz we consider, there is $G^{\bar{\lambda}}=G^{\lambda}$. For detailed discussion, see Appendix **??**. Therefore it does not have a significant impact on subsequent discussions.
2. The vertex would differ by a factor of $-\sqrt{2}$, which is important in kernel calculation.

5.4 Kernel Matrix

Combined the differences mentioned above,(To be more specific, they are the $-\sqrt{2}$ factor in $S(J_{\dots})$ and $1/2$ factor in eqn 32), we can derived the new kernel matrix in the basis of ϕ,ψ,λ,B instead of G. Here, the meaning of $\tilde{k}$ refers to using the k's value in reference [3].

$$\begin{pmatrix} k^{\phi\phi} & k^{\phi\psi} & k^{\phi\lambda} & k^{\phi G} \\ k^{\psi\phi} & 0 & k^{\psi\lambda} & 0 \\ k^{\lambda\phi} & k^{\lambda\psi} & 0 & 0 \\ k^{G\phi} & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \tilde{k}^{\phi\phi} & \tilde{k}^{\phi\psi} & \frac{\tilde{k}^{\phi\lambda}}{2} & \frac{\tilde{k}^{\phi G}}{2} \\ \tilde{k}^{\psi\phi} & 0 & \frac{\tilde{k}^{\psi\lambda}}{2} & 0 \\ 2\tilde{k}^{\lambda\phi} & 2\tilde{k}^{\lambda\psi} & 0 & 0 \\ 2\tilde{k}^{G\phi} & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

5.5 Simple Method in Coefficient

If $E(\Phi)$ and $J(\Phi)$ differ by an arbitrary coefficient, i.e.

$$E(\Phi)=-\sqrt{2}sJ(\Phi)$$

we can consider the change of coefficients in $J_{\dots}$ i.e,

$$J_{ai_1\cdots i_q} \Rightarrow J_{ai_1\cdots i_q} \cdot s$$

Subsequently, by focusing on J^2 and $J_{ai_1\cdots i_q}$, we can adjust the kernel matrix and the coefficients of $n_\lambda n_\phi^q$. For the model $E(\Phi) = J(\Phi)$ we are interested in, $s = -\frac{1}{\sqrt{2}}$

SD Equations:

Notice that each equation contains a J^2 term, so the corresponding modification factor should be s^2, and for the $E(\Phi)$ we are discussing, it is 1/2.

Coefficient of $n_\lambda n_\phi^q$: Notice that it is proportional to $\frac{1}{J^2}$, so the correction factor is $\frac{1}{s^2}$.

Coefficient of the Kernel

The kernel is affected by two parts. The first part is a J^2 contribution as a vertex, so there is a correction factor of s^2 overall. The second part is that the specific expression of the kernel contains $n_\lambda n_\phi^q$, so each element has its own correction factor.

1. $K^{\phi\phi}$ contains two parts, each with a factor of $n_\lambda n_\phi^q$.
2. $K^{\phi\psi}$ contains a factor of $n_\lambda n_\phi^q$.
3. $K^{\phi\lambda}$ contains an unaffected n_ϕ^{q+1}.
4. $K^{\phi G}$ contains an unaffected n_ϕ^{q+1}.
5. $K^{\psi\phi}$ contains a factor of $n_\lambda n_\phi^q$.
6. $K^{\psi\lambda}$ contains an unaffected factor of n_ϕ^{-q-1}.
7. $K^{\lambda\phi}$ contains a factor of $(n_\lambda n_\phi^q)^2$.
8. $K^{\lambda\psi}$ contains a factor of $(n_\lambda n_\phi^q)^2$.
9. $K^{G\phi}$ contains a factor of $(n_B n_\phi^q)^2$.

Coefficient Change of $k^{\Psi\Phi}$ Matrix

$$\begin{bmatrix} 1 & 1 & s^2 & s^2 \\ 1 & 0 & s^2 & 0 \\ \frac{1}{s^2} & \frac{1}{s^2} & 0 & 0 \\ \frac{1}{s^3} & 0 & 0 & 0 \end{bmatrix}$$

It can be observed that the determinant of the characteristic determinant is unchanged when we scale the factor s.

For the model we are discussing, where $E(\Phi) = J(\Phi)$, $s = -\frac{1}{\sqrt{2}}$, so we can obtain the matrix in eqn 33.

6 Summary

In this paper, we introduce two SUSY SYK-like models: the J-type model and the E-type model. The Lagrangians of these two models are defined in very distinct ways. Through calculations, as shown in Chapters 2 and 3, we find that both models yield strikingly similar results for the kernel. We uncover the reason for this phenomenon in the (0,2) L-G model, where we establish that $E(\Phi) = -\sqrt{2}J(\Phi)$ serves as a symmetry condition at the action level. However, more significantly, when examining the characteristic determinant of the kernel matrix, we can relax this condition to $J(\Phi) \propto E(\Phi)$. This implies that a class of models satisfying this proportionality will produce consistent results.

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A Appendix

A.1 The Sign of S_J

If the red negative sign in eqn26 and eqn23 is a positive sign, then the final result would still be

$$\begin{aligned}\mathcal{L} =& \text{kin}_{\phi,\psi,\lambda} + \\ & - 2\bar{J}J - \bar{\psi}^j_+(\sqrt{2}\bar{J}^i_{,j})\bar{\lambda}_i - \lambda_i(\sqrt{2}J^i_{,j})\psi^j_+ \\ & - \bar{E}E - \bar{\psi}^j\bar{E}^i_{,j}\lambda_i - \bar{\lambda}_i E^i_{,j}\psi^j\end{aligned}$$

The only difference is a coefficient in the dual result

$$\boxed{E \Leftrightarrow +\sqrt{2}J, \lambda \leftrightarrow \bar{\lambda}}$$