Feynman Parametrization Note

Leonard Wang

October 18, 2024

Abstract

This is a detailed derivation note primarily based on Chapter 6 of Professor Yu-tin Huang's hep-th paper, arXiv:1210.4226

1 Introduction

This is an unfinished note based on Chapter 6 of Professor Yu-tin Huang's hep-th paper[tHCH12]. I finished the derivation of I_{2tri} and made some effort in deriving following formula, I made some simplification but remaining red words means some puzzles to be solved in the future.

2 Embedding Formalism

Uplift 3-D vector to 5-D adding up constraining condition in order to manifest the symmetry the system has.

Define of null 5 vector :

$y_i := (\vec{x}_i, 1, x_i^2)$

Note that the index i here is note the spacial index, it's a label for different momentum

Define metric and Inner Product For Null Vector:

Convention:$(i \cdot j) := y_i \cdot y_j = g_{MN} y_i^M y_j^N = (x_i - x_j)^2$

We can thus find the metric under the condition that

1. five-dimensional vector V is null vector, which satisfies the condition $V \cdot V = 0$

2.The inner product $(X \cdot Y)$ is equal to $(X - Y)^2$:

$$g_{MN} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Matrix representation of Inner product of $(i \cdot j)$ where the diagonal is zero is because the null embedding formalism and the off-diagonal part which equals to $y_i - y_{i\pm1} = p^2 = 0$ represent on shell condition

$$\eta = \begin{bmatrix} 0 & 0 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & * & * \\ * & 0 & 0 & 0 & * & * & * & * \\ * & * & 0 & 0 & 0 & * & * & * \\ * & * & * & 0 & 0 & 0 & * & * \\ * & * & * & * & 0 & 0 & 0 & * \\ * & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & * & 0 & 0 \end{bmatrix}$$

3 Feynman Para & Cheng-Wu Thm

3.1 Two Part Feynman Para

Here's a integration trick:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[xA + (1-x)B]^2}$$

And here's an example:

$$\frac{1}{(a\cdot i)(a\cdot j)} = \int_0^1 d\alpha_1 \frac{1}{(\alpha_1(a\cdot i) + (1-\alpha_1)(a\cdot i))^2} = \int_0^1 d\alpha_1 d\alpha_2 \frac{\delta(1 - d\alpha_1 - d\alpha_2)}{(\alpha_1(a\cdot i) + \alpha_2(a\cdot i))^2}$$

We set $A = \alpha_1 y_i + \alpha_2 y_j$ numerator becomes $(a\cdot A)^2$. For the δ function part we can use Cheng-Wu theorem to treat it as $\int_0^\infty a_1 a_2 \delta(1-\alpha_1-\alpha_2) = \Gamma[3]\int_0^\infty \frac{da_1 da_2}{\mathrm{GL}(1)}$ 这里的系数有点小问题

there fore,

$$\frac{1}{(a\cdot i)(a\cdot j)} = \Gamma[3]\int_0^\infty \frac{da_1 da_2}{\mathrm{GL}(1)} \frac{1}{(a\cdot A)^2} \tag{1}$$

3.2 N Part Feynman Para

The treatment on δ function should be careful, which is pretty trivial in previous case but not here.

$$\begin{aligned} \frac{1}{A_1\cdots A_n} &= (n-1)!\int_0^1 du_1 \cdots \int_0^{1-u_1} du_2 \cdots \int_0^{1-\sum_{k=1}^{n-1} u_k} du_n \frac{\delta(1-\sum_{k=1}^n u_k)}{\left(\sum_{k=1}^n u_k A_k\right)^n}, \\ &= (n-1)!\int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \times \\ &\quad \frac{1}{[A_1 u_{n-1} + A_2(u_{n-2} - u_{n-1}) + \cdots + A_n(1-u_{n-1})]^n}. \end{aligned} \tag{2}$$

The original definition that can be considered as an integral transform is the second line; hence, the first line is merely a deformation of such an integral transform under the action of the δ function. Under the influence of the δ function, the integral can be expanded to $\int_0^1$.

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 du_1 \cdots du_n \, \delta\left(\sum u_i - 1\right) \frac{(n-1)!}{[u_1 A_1 + u_2 A_2 + \cdots u_n A_n]^n}$$

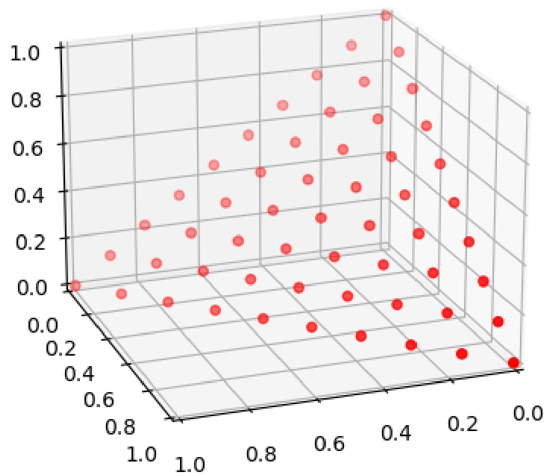


Figure 1: Integration region with δ function constraint

For example, when $n = 3$,

$$\frac{1}{ABC} = \int_0^1 dx\, dy\, dz\, \delta(x + y + z - 1) \frac{2}{[xA + yB + zC]^3}.$$

When using Mathematica for integration, after eliminating the δ function, one should handle the integration intervals carefully, not all variables integrate from $\int_0^1$. The correct handling is as follows:

$$2 \times \int_0^1 \int_0^{1-y_1} \frac{1}{(xA + yB + (1 - x - y)C)^3}\, dy\, dx$$

Intuitively, due to the presence of the δ function, the effective integration region is as shown in the following figure, so the integration region is fixed when there is the δ function.

3.2.1 Schwinger Parametrization:

Other representations of $\frac{1}{AB}$ – Schwinger Parametrization, referenced from [Sch14].

$$\begin{aligned}\frac{1}{AB} &= -\int_0^\infty \tau\, d\tau \int_0^1 dx\, e^{i\tau(A+(B-A)x)} \\ &= \int_0^1 dx \frac{1}{[A + (B - A)x]^2}.\end{aligned}$$

Some common formulas:

$$\frac{1}{A^n B^m} = \frac{\Gamma(m + mn)}{\Gamma(m)\Gamma(n)} \int_0^\infty ds \frac{s^{m-1}}{(A + Bs)^{n+m}}$$

$$\frac{1}{AB} = \int_0^\infty ds \frac{1}{(A + Bs)^2}. \tag{3}$$

For products of multiple a_j variables raised to the power of α_j, the result is [KCL23],

$$\prod_{j=1}^{n}\left(\frac{i}{a_j}\right)^{\alpha_j} = \int_{[0,\infty)} d^n s \prod_{j=1}^{n}\frac{1}{\Gamma[\alpha_j]} s_j^{a_j-1} e^{i a_j s_j} = \int_{[0,\infty)} d^n s\, e^{ia\cdot s}\prod_{i=1}^{n}\frac{s_j^{a_j-1}}{\Gamma[\alpha_j]} \tag{4}$$

where $\int_{[0,\infty)} \mathrm{d}^n s = \prod_{j=1}^{n}\int_0^\infty \mathrm{d}s_j$.

3.3 Cheng-Wu Thm:

[HM19]

3.4 Reproduction of (3.2)

Our goal is to prove following equation,

$$\int_a \frac{\epsilon(a,i,j,k,l)}{(a\cdot i)(a\cdot j)(a\cdot k)(a\cdot l)} - \int dF\ \epsilon(i,j,k,l,\partial_Y)\int_a \frac{\Gamma[3]}{(a\cdot Y)^3}$$

We would use following convention,

$dF := \prod_{i=1}^{4} d\alpha_i \delta(1-\sum_i \alpha_i)$

$Y := \alpha_1 y_i + \alpha_2 y_j + \alpha_3 y_k + \alpha_4 y_l$

By linearity:$(a\cdot Y) = \alpha_1(a\cdot i) + \alpha_2(a\cdot j) + \alpha_3(a\cdot k) + \alpha_4(a\cdot l)$

Derive:

$$\int \frac{-3}{((a\cdot i)\alpha_1 + (a\cdot j)\alpha_2 + (a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^4}\, d\alpha_1 = \frac{1}{(a\cdot i)((a\cdot i)\alpha_1 + (a\cdot j)\alpha_2 + (a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^3}$$

$$\xrightarrow[\alpha_1\to 0]{\text{preserve}} \frac{1}{(a\cdot i)((a\cdot j)\alpha_2 + (a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^3}$$

$$\int \frac{1}{(a\cdot i)((a\cdot j)\alpha_2 + (a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^3}\, d\alpha_2 = -\frac{1}{2(a\cdot i)(a\cdot j)((a\cdot j)\alpha_2 + (a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^2}$$

$$\xrightarrow[\alpha_2\to 0]{\text{preserve}} -\frac{1}{2(a\cdot i)(a\cdot j)((a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^2}$$

$$\int -\frac{1}{2(a\cdot i)(a\cdot j)((a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)^2}\, d\alpha_3 = \frac{1}{2(a\cdot i)(a\cdot j)(a\cdot k)((a\cdot k)\alpha_3 + (a\cdot l)\alpha_4)}$$

$$\xrightarrow[\alpha_3\to 0]{\text{preserve}} \frac{1}{2(a\cdot i)(a\cdot j)(a\cdot k)(a\cdot l)\alpha_4}$$

Using Cheng Wu Theorem, by substituting $\alpha_4 \to 1$ we have integrand I_0

3.5 Single Loop Integral

Our goal is to obtaint (3.4) from (3.3), which is to prove:

$$\Gamma[3]\int_a \frac{1}{(a\cdot Y)^3} := \Gamma[3]\int \frac{d^{D+2}a\delta(a^2)}{i(2\pi)^D \mathrm{Vol(GL(1))}}\frac{1}{(a\cdot Y)^3(a\cdot I)^{D-3}} = \frac{\Gamma\left[3-\frac{D}{2}\right]}{(4\pi)^{\frac{D}{2}}}\frac{1}{(I\cdot Y)^{D-3}(\frac{1}{2}Y^2)^{3-\frac{D}{2}}} \tag{5}$$

$y_I := \left(\vec{0}_D, 0, 1\right)$

Prove:

Using GL(1) Symmetry to gauge fix $(\vec{a}, a^{(D+1)}, a^{(D+2)})$ to $(\vec{a}, 1, a^{(D+2)})$

Since $\delta(a^2) = \delta(a \cdot a) = \delta(g_{MN} a^M a^N)$, therefore a^{D+2} has to be $\vec{a} \cdot \vec{a}$

Thus $\int_a = \int \frac{d^D x}{i(2\pi)^D}$ where x is the component of a and we can use Loop Integral Formula [PS95]

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}}$$

First lets deal with $(a \cdot Y)$ Notice that inner product of Y which is $Y \cdot Y := Y^2$ is no longer zero along with a. Therefore we can not write $(a \cdot Y)$ as $[\vec{x_a} - (\sum_{i=1}^{4} \alpha_i \vec{y_i})]^2$, which only defines for null vector inner product. Instead we have to write it as $\alpha_1(a \cdot i) + \alpha_2(a \cdot j) + \alpha_3(a \cdot k) + \alpha_4(a \cdot l)$ which is

$$\alpha_1(x_a - y_i)^2 + \alpha_2(x_a - y_j)^2 + \alpha_3(x_a - y_k)^2 + \alpha_4(x_a - y_l)^2$$

we can be written as

$$= \sum_{i=1}^{4} \alpha_i \times \left(a^2 - \frac{\sum_{i=1}^{4} 2\alpha_i(\vec{a} \cdot \vec{y})}{\sum_{i=1}^{4} \alpha_i}\right)$$

$$= \sum_{i=1}^{4} \alpha_i \times \left(a - \frac{\sum_{i=1}^{4} \alpha_i y_i}{\sum_{i=1}^{4} \alpha_i}\right)^2 - \left(\frac{\sum_{i=1}^{4} \alpha_i y_i}{\sum_{i=1}^{4} \alpha_i}\right)^2$$

$$= (a - Y)^2 - Y^2 \tag{6}$$

the last step is because $\sum_{i=1}^{4} \alpha_i = 1$.

So we can identify $\vec{l}$ as $(\vec{a} - \vec{Y})$ meanwhile $d^D l = d^D a$ and Δ as Y^2, and $n = 3$

And we can get $\int \frac{d^D x}{i(2\pi)^D} \frac{1}{(((a-Y)^2 - Y^2)^3) \cdot (a \cdot I)^{D-3}}$

Here, $a \cdot I$ and $I \cdot Y$ are just factors used to balance the conformal weight. When incorporated into the metric we obtain, both contribute a value of 1.

4 Feynman Parametrization In 2 Loop

Given Formula

$$\begin{aligned}
I_{2tri}^{i,i+2;i+2,i} &:= \int_{a,b} \frac{(i \cdot i+2)^2}{(a \cdot i)(a \cdot i+2)(a \cdot b)(b \cdot i)(b \cdot i+2)} \\
I_{2tri}^{i,i+2;i-2,i} &:= \int_{a,b} \frac{(i \cdot i+2)(i \cdot i-2)}{(a \cdot i)(a \cdot i+2)(a \cdot b)(b \cdot i-2)(b \cdot i)} \\
I_{2tri}^{i,i+2;i-3,i-1} &:= \int_{a,b} \frac{(i \cdot i+2)(i-1 \cdot i-3)}{(a \cdot i)(a \cdot i+2)(a \cdot b)(b \cdot i-3)(b \cdot i-1)}
\end{aligned} \tag{7}$$

4.1 I_{2tri} Calculation

$I_{2tri}^{1,3;3,1}$

Using formula in (9) and substituting i to 1

$$I_{2tri}^{1,3;3,1} := \int_{a,b} \frac{(1\cdot 3)^2}{(a\cdot 1)(a\cdot 3)(a\cdot b)(b\cdot 1)(b\cdot 3)} \tag{8}$$

Using formula (1) for loop variable a and b we obtain

$$I_{2tri}^{1,3;3,1} = \Gamma[3]^2 \int_0^\infty \frac{[d^1 a_1 a_3]}{\mathrm{vol(GL(1))}} \frac{[d^1 b_1 b_3]}{\mathrm{vol(GL(1))}} \int_{a,b} \frac{(1.3)^2}{(a\cdot A)^2 (a\cdot b)(b\cdot B)^2}. \tag{9}$$

with $A = \sum_{i=1.3} a_i y_i$ and $B = \sum_{i=1.3} b_i y_i$.

For start, let's dealing with first part $\int_a \frac{1}{(a\cdot A)^2 (a\cdot b)}$. Using a integral transformation similar to α parametrization $\int\limits_0^\infty dx \frac{1}{(A+Bx)^3} = \frac{2}{A^2 B}$ we can write it as

$$\Gamma(3) \int_0^\infty df \int_a \frac{1}{(a\cdot (A+fb))^3}$$

More than that, we can use loop integral formula (5) with $D=3$ we get

$$\int df \frac{\Gamma\left[3-\frac{3}{2}\right]}{(4\pi)^{\frac{3}{2}}} \frac{1}{(I\cdot (A+fb))^{3-3}(\frac{1}{2}(A+fb)^2)^{3-\frac{3}{2}}} = \int df \frac{1}{16\pi} \frac{1}{(\frac{1}{2}(A+fb)\cdot (A+fb))^{3/2}}$$

We would ask b^2 =0 (since all single 5- vector which satisfy $(\vec{v},1,v^2)$ are null under our metric) and we get

$$\frac{1}{4\pi} \times \int_0^\infty df \frac{1}{4} \frac{1}{\frac{1}{2}(A^2+2Abf+b^2f^2)^{3/2}} \xrightarrow{b^2=0} \frac{1}{4\pi} \times \frac{1}{4} \int_0^\infty df \frac{1}{\frac{1}{2}(A^2+2Abf)^{3/2}}$$

.

After integration of f we have

$$\frac{1}{2} \frac{1}{\sqrt{\frac{1}{2}A\cdot A}(A\cdot b)}$$

- In the paper they would strip the $\frac{1}{4\pi}$ for each loop for convenience.

These derivation means $\int_a \frac{1}{(a\cdot A)^2 (a\cdot b)} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{2}A\cdot A}(A\cdot b)}$ therefore

$$\int_{a,b} \frac{1}{(a\cdot A)^2 (a\cdot b)(b\cdot B)^2} = \frac{1}{2} \int_b \frac{1}{\sqrt{\frac{1}{2}A\cdot A}(A\cdot b)(b\cdot B)^2}$$

Same transformation as above (By omitting $1/4\pi$) $\int_a \frac{1}{(a\cdot A)^2(a\cdot b)} \simeq \frac{1}{4} \int_0^\infty df \; \frac{1}{(\frac{1}{2}(A+fb)\cdot (A+fb))^{3/2}}$ we get

$$\frac{1}{2\sqrt{\frac{1}{2}A\cdot A}} \int_a \frac{1}{(A\cdot b)\,(b\cdot B)^2} \simeq \frac{1}{8} \frac{1}{\sqrt{\frac{1}{2}A\cdot A}} \int_0^\infty de \; \frac{1}{(\frac{1}{2}(B+eA)\cdot (B+eA))^{3/2}}$$

Using integration trick $\frac{1}{4\pi} \int_0^\infty dc \frac{1}{\sqrt{c}(Xc+Y)^2} = \frac{1}{8\sqrt{X}Y^{3/2}}$ we can write above integration as

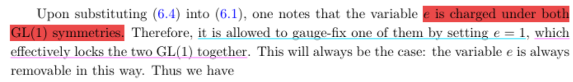

Upon substituting (6.4) into (6.1), one notes that the variable e is charged under both GL(1) symmetries. Therefore, it is allowed to gauge-fix one of them by setting $e = 1$, which effectively locks the two GL(1) together. This will always be the case: the variable e is always removable in this way. Thus we have

Figure 2: Enter Caption

$$\int_0^\infty \frac{dc}{4\pi\sqrt{c}} \int_0^\infty de \frac{1}{(c\frac{1}{2}A\cdot A + \frac{1}{2}(eA+B)\cdot(eA+B))^2}$$

• Rmk: Since $A^2 \neq 0$ we can not use same procedure to reduce this integral.

4.1.1 Back to Integration (6.1)

Substitute our integral transformation in (6.1) and get

$$\Gamma(3)^2 \int_0^\infty \frac{[d^1 a_1 a_3]}{\mathrm{vol(GL(1))}} \frac{[d^1 b_1 b_3]}{\mathrm{vol(GL(1))}} \int_0^\infty \frac{dc}{4\pi\sqrt{c}} \int_0^\infty de \frac{1}{(c\frac{1}{2}A\cdot A + \frac{1}{2}(eA+B)\cdot(eA+B))^2}$$

We can gauge fix e as 1. We can see that e is coupled to both A and B , we can remove our gauge fixing on a_i and b_i to e. For example, when we remove the gauge from b_i, it becomes $[d^2 b_1 b_3]$ together with a_i we have $[d^3 a_1 a_3 b_1 b_3]$

不知道我的理解正不正确

$$I_{2tri}^{1331} = \int_0^\infty \frac{dc}{4\pi\sqrt{c}} \int \frac{[d^3 a_1 a_3 b_1 b_3]}{\mathrm{vol(GL(1))}} \frac{(1\cdot 3)^2}{\left((1+c)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B\right)^2}.$$

Now we need to deal with regularization, and all we need to do is to shift $(i\cdot j) \to x_{ij}^2 + 2\mu_{IR}^2$

• Rmk we have to be careful with those $(i\cdot j)$ with $i = j$, which was previously dropped as 0 but preserved in the regularization. The ordering of regularization should be watched out for. And we only need to consider the leading behaviour of numerator so we can leave the regularization away from the numerator.

4.1.2 Regularization

We define $\epsilon := \frac{\mu_{IR}^2}{x_{13}^2}$ and we simplify the inner product on denominator,

$$\begin{aligned}
&\frac{a_1^2(1\cdot 1)}{2} + a_1 b_1 (1\cdot 1) + \frac{b_1^2(1\cdot 1)}{2} + \frac{1}{2}a_1^2 c(1\cdot 1) \\
&+ a_1 a_3 (1\cdot 3) + a_3 b_1 (1\cdot 3) + a_1 b_3 (1\cdot 3) + b_1 b_3 (1\cdot 3) \\
&+ a_1 a_3 c(1\cdot 3) + \frac{a_3^2(3\cdot 3)}{2} + a_3 b_3 (3\cdot 3) + \frac{b_3^2(3\cdot 3)}{2} \\
&+ \frac{1}{2}a_3^2 c(3\cdot 3)
\end{aligned}$$

We do the regularization and divide the numerator, $(i,i) \to \epsilon, (1,3) \to 1+\epsilon$ and we get,

$$\begin{aligned}
&a_1^2\epsilon + a_3^2\epsilon + 2a_1 b_1\epsilon + b_1^2\epsilon + 2a_3 b_3\epsilon + b_3^2\epsilon + a_1^2 c\epsilon + a_3^2 c\epsilon \\
&+ a_1 a_3(1+2\epsilon) + a_3 b_1(1+2\epsilon) + a_1 b_3(1+2\epsilon) \\
&+ b_1 b_3(1+2\epsilon) + a_1 a_3 c(1+2\epsilon)
\end{aligned}$$

which is exactly what was shown in the paper.

Change Variable Observing the homogeneous of denominator, we can do the variable change by substitute the variables as

$$a_1 = 1, \quad b_1 = x, \quad a_3 = a, \quad b_3 = ay, \quad \frac{[d^3 a_1 a_3 b_1 b_3]}{\mathrm{vol}(\mathrm{GL}(1))} = adadxdy$$

We gauge fix a_1 as 1 and a_1, b_1 shares the same character in the integrand denominator, so does a_3, b_3 So we may use x, y to represent their behavior,. Last but not least, we need a variable a to describe the scaling behavior of those variables.

And there fore the denominator in side the square is

$$\frac{a}{\left(a^2\varepsilon\left(c+(y+1)^2\right)+a(2\varepsilon+1)(c+xy+x+y+1)+\varepsilon\left(c+(x+1)^2\right)\right)^2}$$

If we replace a in to $\frac{1}{p}$ and change relevant integration measure, the integrand becomes

$$-\frac{p}{\left(p^2\varepsilon\left(c+(x+1)^2\right)+p(2\varepsilon+1)(c+xy+x+y+1)+\varepsilon\left(c+(y+1)^2\right)\right)^2}$$

The minus sign would be cancelled with the inversion of integral interval and get $\int_0^\infty$ in the end. Since we can do the inversion, we only need to consider the integration region of $[1, \infty]$ and multiply by 2. In that region the term $a\epsilon$ is suppressed by ϵ which can be thrown away when we do the integration over a

$$\begin{aligned} &a^2c\epsilon + a^2y^2\epsilon + 2a^2y\epsilon + a^2\epsilon + 2ac\epsilon + ac \\ &\quad + 2axy\epsilon + axy + 2ax\epsilon + ax + 2ay\epsilon + ay + 2a\epsilon + a + c\epsilon + x^2\epsilon + 2x\epsilon + \epsilon \end{aligned}$$

After throwing all therm include $a\epsilon$

$$ac + axy + ax + ay + a + c\epsilon + x^2\epsilon + 2x\epsilon + \epsilon$$

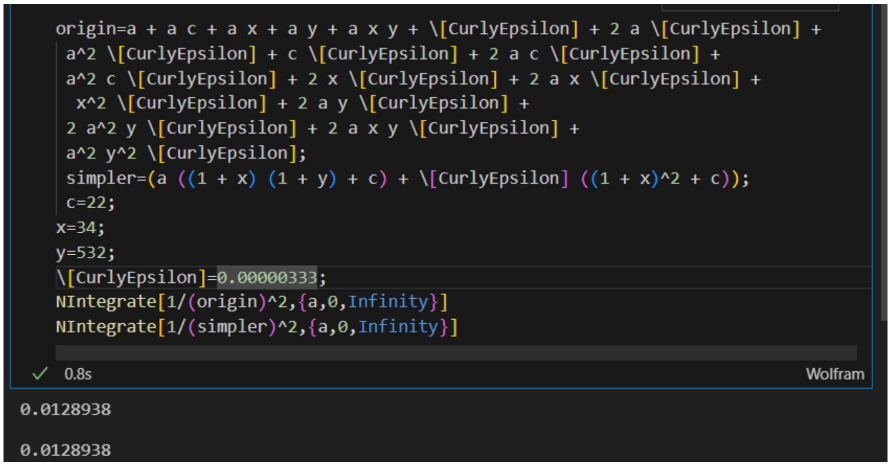
Although it sees this operation messed up the inversion property of the integrand, this operation still applies in the limit $\lim \epsilon \to 0$ and we can prove this numerically. The result shown in fig 3 reveals that even though we thrown away some term, integration over $[0, \infty]$ get same result. In previous analysis, we see those integrand are same in the region of $[1, \infty]$ so we can conclude that the integration are same in the region of $[0, 1]$ which means the inversion symmetry are preserved numerically. Since the integration over $[1, \infty]$ gives the form of $\frac{\frac{B}{aA+B}+\log(aA+B)}{A^2}$ which is divergent, we can integrate on the dual region which is $[0, 1]$

4.2 Derive (6.6) and (6.7)

(6.6)

Integration that we have discussed above is written as

$$I_{2tri}^{13;31} = 2\int_0^\infty \frac{dc}{4\pi\sqrt{c}} \int_0^1 ada \int_0^\infty \frac{dxdy}{(a((1+x)(1+y)+c)+\epsilon((1+x)^2+c))^2}$$



```
origin=a + a c + a x + a y + a x y + \[CurlyEpsilon] + 2 a \[CurlyEpsilon] +
 a^2 \[CurlyEpsilon] + c \[CurlyEpsilon] + 2 a c \[CurlyEpsilon] +
 a^2 c \[CurlyEpsilon] + 2 x \[CurlyEpsilon] + 2 a x \[CurlyEpsilon] +
  x^2 \[CurlyEpsilon] + 2 a y \[CurlyEpsilon] +
 2 a^2 y \[CurlyEpsilon] + 2 a x y \[CurlyEpsilon] +
 a^2 y^2 \[CurlyEpsilon];
 simpler=(a ((1 + x) (1 + y) + c) + \[CurlyEpsilon] ((1 + x)^2 + c));
 c=22;
x=34;
y=532;
\[CurlyEpsilon]=0.00000333;
NIntegrate[1/(origin)^2,{a,0,Infinity}]
NIntegrate[1/(simpler)^2,{a,0,Infinity}]
0.8s  Wolfram
0.0128938

0.0128938
```

Figure 3: Verify inversion symmetry numerically

First we do the integration over a. Since $\int_0^1 da \frac{a}{(aA+B)^2} = \frac{-\frac{A}{A+B}+\log(A+B)-\log(B)}{A^2}$, and B here is $\epsilon((1+x)^2+c)-1$, while $\epsilon \ll 1$ we can throw away this part in the final result and the leading term is $\frac{(\log(\frac{A}{B})-1)}{A^2}$ and therefore what we get is

$$2\int \frac{dc}{4\pi\sqrt{c}} \int_0^\infty dxdy \frac{\log\frac{(1+x)(1+y)+c}{\epsilon((1+x)^2+c)}-1}{((1+x)(1+y)+c)^2} + \mathcal{O}(\epsilon)$$

We can separate the ϵ part along and make the integration to see the dependence on μ_{IR} and

$$\int_0^\infty dx \frac{\log(\epsilon)}{(c+(x+1)(y+1))^2} = -\frac{\log(\epsilon)}{(y+1)(c+y+1)}$$

$$\int_0^\infty dy \frac{\log(\epsilon)}{(y+1)(c+y+1)} = -\frac{\log(c+1)\log(\epsilon)}{c}$$

$$-2\int_0^\infty dc \frac{\log(c+1)\log(\epsilon)}{4\pi c\sqrt{c}} = -\log(\epsilon)$$

Other coefficient can be calculated by mathematica.

(6.7)

From eqn (7) we have

$$I_{2tri}^{13;35} = \int_{a,b} \frac{(1\cdot 3)(3\cdot 5)}{(a\cdot 1)(a\cdot 3)(a\cdot b)(b\cdot 3)(b\cdot 5)}$$

First we'll do feynman parametrization, and we get

$$I_{2tri}^{1,3;3,5} = \Gamma[3]^2 \int_0^\infty \frac{[d^1 a_1 a_3]}{\text{vol(GL(1))}} \frac{[d^1 b_3 b_5]}{\text{vol(GL(1))}} \int_{a,b} \frac{(1\cdot 3)(3\cdot 5)}{(a\cdot A)^2(a\cdot b)(b\cdot B)^2}$$

with $A = \sum_{i=1.3} a_i y_i$ and $B = \sum_{i=3,5} b_i y_i$.

We use exact integral transformation trick mentioned above and

$$\int_{a,b} \frac{1}{(a\cdot 1)(a\cdot 3)(a\cdot b)(b\cdot 3)(b\cdot 5)} =$$
$$\Gamma(3)^2 \int_0^\infty \frac{[d^1 a_1 a_3]}{\text{vol}(\text{GL}(1))} \frac{[d^1 b_3 b_5]}{\text{vol}(\text{GL}(1))} \int_0^\infty \frac{dc}{4\pi\sqrt{c}} \int_0^\infty de \frac{1}{(c\frac{1}{2}A\cdot A + \frac{1}{2}(eA+B)\cdot(eA+B))^2} \tag{10}$$

And well use same gauge fixing procedure as above and get

$$I_{2tri}^{13;35} = \int_0^\infty \frac{dc}{4\pi\sqrt{c}} \int \frac{[d^3 a_1 a_3 b_3 b_5]}{\text{vol}(\text{GL}(1))} \frac{(1\cdot 3)(3\cdot 5)}{\left((1+c)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B\right)^2}$$

不知道怎么变换过去的

After regularization, the integrand can be written as

$$\frac{1}{((a_1+b_5)(a_3+b_3) + a_1 b_5 + c a_1 a_3 + \epsilon'((a_1+a_3+b_3+b_5)^2 + c(a_1+a_3)^2)^2}$$

对 collinear region 的处理

We identify the dangerous region as the collinear region $a_1 \to 0$ and $b_1 \to 0$ and therefore we can drop the term a_1, b_5 multiplied by ϵ' and

Now we'll do the regularization

$$\text{integrand} = \frac{1}{\left(a_1(a_3 c + a_3 + b_3 + b_5) + a_3^2(c+1)\epsilon' + a_3(2b_3\epsilon' + b_5) + b_3(b_3\epsilon' + b_5)\right)^2}$$

$$\int_0^\infty \text{integrand} db_5$$
$$= -\frac{1}{(a_1+a_3+b_3)\left(a_1(a_3 c + a_3 + b_3) + a_3^2(c+1)\epsilon' + a_3(2b_3\epsilon') + b_3(b_3\epsilon')\right)}$$

After gauge fixing $a_3 = 1$ and integrate $a_1$we have

$$\int_0^\infty \% da_1 = -\frac{-\log\left(\epsilon'\left((b_3+1)^2+c\right)\right) + \log(b_3+c+1) + \log(b_3+1)}{\epsilon'\left((b_3+1)^2+c\right) - (b_3+1)(b_3+c+1)}$$

Since $\epsilon' \ll 1$ we can rewrite it as

$$\frac{\log((b_3+1)(b_3+c+1)) - \log\left((b_3+1)^2+c\right) - \log(\epsilon')}{(b_3+1)(b_3+c+1)}$$

For the integration dependence on ϵ' we have the following

$$-\int_0^\infty db_3 \frac{\log(\epsilon')}{(b_3+1)(b_3+c+1)} = -\frac{\log(c+1)\log(\epsilon')}{c}$$
$$-\int_0^\infty dc \frac{\log(c+1)\log(\epsilon')}{4\pi c\sqrt{c}} = -\frac{\log(\epsilon')}{2}$$

Other coefficient can be obtained by calculation on mathematica and we 'll ultimately have $1 - \frac{1}{2}\log 4\epsilon' + O(\mu_{\text{IR}})$ as expected.

5 Contraction of $\epsilon(*,*,*,*,*)$ and Related Operations

5.1 1.0 Basic Formulas

Gram determinant formula (3.16)

$$\epsilon(i_1,\ldots,i_5)\epsilon(j_1,\ldots,j_5) := \det\left[(i_i \cdot j_j)\right]$$

$$\epsilon(a,i,j,k,*)\epsilon(b,l,m,n,*) := \epsilon(a,i,j,k,{}^{\mu})\epsilon(b,l,m,n,{}_{\mu})$$

Refer[CH11]

In the Embedding Formalism,

$$X_i = \begin{pmatrix} \frac{1}{2}x_i^2 \\ 1 \\ \vec{x_i} \end{pmatrix}$$

$$\epsilon(5,1,2,3,4) = \begin{vmatrix} \frac{1}{2}x_5^2 & \frac{1}{2}x_1^2 & \frac{1}{2}x_2^2 & \frac{1}{2}x_3^2 & \frac{1}{2}x_4^2 \\ 1 & 1 & 1 & 1 & 1 \\ \vec{x}_5 & \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{vmatrix}$$

5.2 1.1 Derivation of Formula 6.9

Regarding the derivation of

$$\epsilon(\partial_A,1,2,3,*)\epsilon(\partial_B,4,5,6,*)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^2}$$

The first step is

$$-2\epsilon(\partial_A,1,2,3,*)\epsilon((A+B),4,5,6,*)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3}$$

The second step is?

$$\begin{aligned} &-2\epsilon(1,1,2,3,*)\epsilon((A+B),4,5,6,*)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3} \\ &+6\epsilon((B+A(1+c)),1,2,3,*)\epsilon((A+B),4,5,6,*)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3} \end{aligned}$$

Expression given by Jia Kai Guo

$$\frac{6\begin{vmatrix} (c+1)((A\cdot A)+(A\cdot B))+(B\cdot A)+(B\cdot B) & (1\cdot A)+(1\cdot B) & (2\cdot A)+(2\cdot B) & (3\cdot A)+(3\cdot B) \\ (c+1)(A\cdot 4)+(B\cdot 4) & (1\cdot 4) & (2\cdot 4) & (3\cdot 4) \\ (c+1)(A\cdot 5)+(B\cdot 5) & (1\cdot 5) & (2\cdot 5) & (3\cdot 5) \\ (c+1)(A\cdot 6)+(B\cdot 6) & (1\cdot 6) & (2\cdot 6) & (3\cdot 6) \end{vmatrix}}{\left((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B\right)^4}$$

$$-\frac{2\times 2\begin{vmatrix} (1\cdot 4) & (2\cdot 4) & (3\cdot 4) \\ (1\cdot 5) & (2\cdot 5) & (3\cdot 5) \\ (1\cdot 6) & (2\cdot 6) & (3\cdot 6) \end{vmatrix}}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3}$$

The current issue is how to get from

$$2\epsilon(\partial_A, 1, 2, 3, *)\left(\epsilon((A+B), 4, 5, 6, *)\right)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3}$$

I guess it should be equal to

$$2\epsilon(1, 1, 2, 3, *)\epsilon(1, 4, 5, 6, *)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3}$$

I'm not sure if this is equal to

$$2\epsilon(*, 1, 2, 3, *)\epsilon(*, 4, 5, 6, *)\frac{1}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3}$$

Still hard to get the right coefficient

$$\frac{2\times 2\begin{vmatrix} (1\cdot 4) & (2\cdot 4) & (3\cdot 4) \\ (1\cdot 5) & (2\cdot 5) & (3\cdot 5) \\ (1\cdot 6) & (2\cdot 6) & (3\cdot 6) \end{vmatrix}}{((c+1)\frac{1}{2}A\cdot A + A\cdot B + \frac{1}{2}B\cdot B)^3}$$

6 Detailed integration

6.1 Formula 6.13

$$-\int_0^\infty \frac{dc}{4\pi\sqrt{c}}\int_{a_1<b_1, a_3<b_3}\frac{[d^4a_1a_2a_3b_1b_3]}{\text{vol}(\text{GL}(1))}\frac{a_2+2b_1+2b_3}{(a_2+b_1+b_3)^2(b_1b_3+a_1a_3c+\varepsilon a_2^2(1+c))^2}$$

1. Fix the gauge by setting a_2 to maintain symmetry.
2. Integrate over a_1 and a_3 in the integrand, which does not contain c, to obtain the result:

$$\frac{(2b_1+2b_3+1)(\log((c+1)(b_1b_3+\varepsilon)) - \log(b_1b_3+c\varepsilon+\varepsilon))}{c(b_1+b_3+1)^2(b_1b_3+c\varepsilon+\varepsilon)}$$

3. Integrate over c to get a good result:

$$\frac{(2b_1+2b_3+1)\left(2\sqrt{b_1b_3+\varepsilon}+\sqrt{\varepsilon}\left(\log\left(-\frac{4(\sqrt{\varepsilon}-\sqrt{b_1b_3+\varepsilon})^2}{(\sqrt{b_1b_3+\varepsilon}+\sqrt{\varepsilon})^2}\right)-2\right)\right)}{4(b_1+b_3+1)^2(b_1b_3+\varepsilon)^{3/2}}$$

4. However, it is unclear how to proceed further.

6.2 Formula 6.14

$$I^{\text{col}}(y) := \int_0^\infty \frac{dc}{4\pi\sqrt{c}}\int_{a_1<b_1}\frac{[d^2a_1a_2b_1]}{\text{vol}(\text{GL}(1))}\frac{y(a_2y+2b_1)\log\left(\frac{(a_2+b_1)^2+c(a_1+a_2)^2}{a_2^2(1+c)}\right)}{b_1(b_1+a_1c)(a_2y+b_1)^2} = \frac{\pi^2}{6} - \text{Li}_2(1-y)$$

We choose to fix the gauge by setting a_2. Then, the inner integral can be written as:

$$\int_{a_1<b_1} d^2a_1b_1\frac{y(y+2b_1)\log\left(\frac{(1+b_1)^2+c(a_1+1)^2}{1+c}\right)}{b_1(b_1+a_1c)(y+b_1)^2}$$

The result of integrating over a_1 is:

$$\frac{y(2b_1+y)}{b_1 c(b_1+y)^2}\Bigg(-\mathrm{Li}_2\left(-\frac{b_1\left(c+b_1(\sqrt{-c}-1)+\sqrt{-c}\right)}{b_1^2+c}\right)$$
$$+\log(c+1)\left(\log(b_1^2+c)-\log(c)\right)$$
$$+\mathrm{Li}_2\left(\frac{b_1}{\sqrt{-c}b_1+b_1-c+\sqrt{-c}}\right)$$
$$-\mathrm{Li}_2\left(\frac{b_1\sqrt{-c}(c+1)}{(-c)^{3/2}+c+b_1(c+\sqrt{-c})}\right)$$
$$+\mathrm{Li}_2\left(\frac{b_1\sqrt{-c}}{(-c)^{3/2}+c+b_1(c+\sqrt{-c})}\right)\Bigg)$$

Using the formula $-\int dt \frac{x\log(t)}{1-tx} = \mathrm{Li}_2(tx)+\log(t)\log(1-tx)$ to handle the Li_2 terms, we get the expression:

$$\frac{y(2b_1+y)\left(\log(c+1)\left(\log\left(b_1^2+c\right)-\log(c)\right)-\frac{2b_1 c\log(t)\left(b_1^3(t(c-t+2)-1)+b_1^2 c\left(t^2+1\right)+b_1 c((c+2)t-1)+c^2\right)}{\left(b_1^2(c+(t-1)^2)+2b_1 ct+c^2+c\right)\left(b_1^2(t(ct+t-2)+1)+2b_1 ct+c\right)}\right)}{b_1 c(b_1+y)^2}$$

However, subsequent integrals over b_1 or c are difficult to handle.

6.3 Formula 6.19

We need to verify the integral:

$$I_{\text{box,tri}}^{\text{odd}}(1) = \int_0^\infty \frac{dc}{4\pi\sqrt{c}}\frac{\log\frac{u_2}{u_3}\log(u_1(c+1))}{1-u_1(c+1)} = \frac{\log\frac{u_3}{u_2}\arccos(\sqrt{u_1})}{2\sqrt{u_1(1-u_1)}}$$

Substituting the integral into Mathematica 13 yields the result:

$$\frac{(\log(u_2)-\log(u_3))((\log(-\sqrt{u_1}+i\sqrt{1-u_1})-i\pi)(\log(c)-\log(1-u_1))}{8\pi\sqrt{-((u_1-1)u_1)}}$$
$$+\frac{\arccos(\sqrt{u_1})(i\log(c)-i\log(1-u_1)-4\pi)}{8\pi\sqrt{-((u_1-1)u_1)}}$$

Using the identity $\arccos x = -i\ln\left(x+\sqrt{x^2-1}\right)$, and considering that the first part contains $i\arccos$, we retain the real part of the result as follows:

$$-\frac{\arccos(\sqrt{u_1})(\log(u_2)-\log(u_3))}{2\sqrt{-((u_1-1)u_1)}}$$

7 Appendix

7.1 Properties Related to $\mathrm{Li}_2(x)$

Definition:

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2} = \int_0^x \frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{n} = \int_x^0 \frac{\ln(1-t)}{t}\mathrm{d}t$$

Common Formulas:

(1) $\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\ln^2(1-x)$ $(x < 1)$;

(2) $\mathrm{Li}_2(x^2) = 2\,\mathrm{Li}_2(x) + 2\,\mathrm{Li}_2(-x)$;

(3) $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x)$.

(4) $\mathrm{Li}_2(-1) = -\frac{1}{12}\pi^2$

(5) $\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)$

Reference: https://mathworld.wolfram.com/Dilogarithm.html

7.2 Useful Mathematica Commands

7.2.1 $\mathrm{Li}_2(x)$

The general $\mathrm{Li}_2(x)$ is difficult to integrate with other functions; consider using integration by parts, but the following method is more general:

```
****/. PolyLog[2, x_] :> (-x Log[t])/(1 - t x) // FullSimplify;
Assuming[** \[Element] PositiveReals && 1 > t > 0, ****];

**** // Expand;
Assuming[t \[Element] PositiveReals, Integrate[List @@ %, t]];
Normal[Assuming[u1 \[Element] PositiveReals && 0 > t > -1,
    Series[% /. t -> 1 + t, {t, 0, 0}]]] /. Floor[__] -> 0;
Normal[Assuming[u1 \[Element] PositiveReals && 1 > t > 0,
    Series[%%, {t, 0, 0}]]] /. Floor[__] -> 0;
Result = %% - % // Total
```

The mathematical basis for this method is

$$-\int dt \frac{x\log(t)}{1-tx} = \mathrm{Li}_2(tx) + \log(t)\log(1-tx)$$

Finally, integrating term by term can improve computation speed; generally, after introducing t, the series expansion yields many terms.

Maple cannot read the HyperInt.mpl file; may need to try an older version.

7.2.2 Numerical Verification of Integrals Using Random Numbers

```
temp = %;
Thread[Variables[temp /. Log -> Plus] ->
  RandomReal[20, Length[Variables[temp /. Log -> Plus]]]]
NIntegrate[****/. %, {**, 0, Infinity}]
temp /. %%
```

Not expanding ‘Log‘ will recognize additional variables.

7.2.3 Integration trick

Do the indefinite integral and treat the upper limit as $1/x$, the integral over $\int_0^\infty$ can be represented by expansion.

Example code of integration of a_1 over function I_{A2}

```
    Integrate[IA1, a1];
Assuming[a1 \[Element] PositiveReals && a2 \[Element] PositiveReals &&
    a3 \[Element] PositiveReals && b4 \[Element] PositiveReals &&
   b5 \[Element] PositiveReals && b6 \[Element] PositiveReals &&
   c \[Element] PositiveReals && u1 \[Element] PositiveReals &&
   u2 \[Element] PositiveReals && u3 \[Element] PositiveReals &&
   u5 \[Element] PositiveReals,
  Normal[Series[% /. a1 -> 1/a1, {a1, 0, 0}]]];
Assuming[a1 \[Element] PositiveReals && a2 \[Element] PositiveReals &&
    a3 \[Element] PositiveReals && b4 \[Element] PositiveReals &&
   b5 \[Element] PositiveReals && b6 \[Element] PositiveReals &&
   c \[Element] PositiveReals && u1 \[Element] PositiveReals &&
   u2 \[Element] PositiveReals && u3 \[Element] PositiveReals &&
   u5 \[Element] PositiveReals, Normal[Series[%%, {a1, 0, 0}]]];
IA2 = %% - % // Simplify
```

8 Acknowledgement

Thanks professor Kai Yan's detailed instruction and Dr JiaKai Guo's discussion.

References

[CH11] Wei-Ming Chen and Yu-tin Huang. Dualities for loop amplitudes of n = 6 chern-simons matter theory. *Journal of High Energy Physics*, 2011(11), November 2011.

[HM19] Martijn Hidding and Francesco Moriello. All orders structure and efficient computation of linearly reducible elliptic feynman integrals, 2019.

[KCL23] U-Rae Kim, Sungwoong Cho, and Jungil Lee. The art of Schwinger and Feynman parametrizations. *J. Korean Phys. Soc.*, 82(11):1023–1039, 2023.

[PS95] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[Sch14] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3 2014.

[tHCH12] Yu tin Huang and S. Caron-Huot. The two-loop six-point amplitude in abjm theory, 2012.