Field Theory in Cosmology

§1.4 Single Field Inflation $P(X,\phi)$ Theory

Scalar field couple to gravity $P(X,\phi)$ Theory.

$$\boxed{S=\int\sqrt{-g}\left[\frac{1}{2}M_{pl}^2R+P(X,\phi)\right]}$$

P.X theory
k-inf
k-essence

$P(X,\phi)=P(X)$
↓
Superfluid.

- X 代表正则动能

  $X=-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi=\frac{1}{2}\left[\dot\phi^2-(\partial_i\phi)^2\right]$
- 正则标量场的 P: $P=X-V(\phi)$
- 上述 scalar-field 和引力是 minimally coupled.

- 几个相关量
  - $T_{\mu\nu}=P_{,X}\partial_\mu\phi\partial_\nu\phi+g_{\mu\nu}P(X,\phi)$

    推导: $T_{\mu\nu}=\frac{-2}{\sqrt{-g}}\cdot\frac{\delta S}{\delta g^{\mu\nu}}$

    $g_{,u}=g\cdot g^{\alpha\beta}g_{\alpha\beta,u}\Rightarrow\delta g=g^{\alpha\beta}\delta g_{\alpha\beta}\cdot g=-\delta g^{\alpha\beta}g_{\alpha\beta}\cdot g$

    $\delta S=\int\sqrt{-g}\,-\frac{1}{2}\delta g^{\mu\nu}g_{\mu\nu}P(X,\phi)-\frac{1}{2}\delta g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi P_{,X}$

    而 $\delta S=-\int\frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu}$

    $=\int\sqrt{-g}\,\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu}$

    $\therefore T_{\mu\nu}=g_{\mu\nu}P(X,\phi)+\partial_\mu\phi\partial_\nu\phi P_{,X}$
  - $\rho=2X\cdot P_{,X}-P$ [认为 $\partial_i\phi=0$ 时]　　∴ p 是压强. 大 P 是 $P(X,\phi)$

    $p=P$

    $u_\mu=-\frac{\partial_\mu\phi}{\sqrt{2X}}$　　$[T_{\mu\nu}=(\rho+P)u_\mu u_\nu+g_{\mu\nu}P$　　homogeneous perfect fluid].

    按 $T^\mu{}_\nu=\mathrm{Diag}\{-\rho,p,p,p\}\Rightarrow T_{\mu\nu}=\mathrm{Diag}\{\rho,a^2p,a^2p,a^2p\}$
  - 背景场:　认为 background 是 homogeneous 的 $\phi=\bar\phi(t)$

$$\boxed{\ddot\phi\,(P_{,X}+2XP_{,XX})+3H\dot\phi P_{,X}+(2XP_{,X\phi}-P_{,\phi})=0}$$

De Sitter:

$$a=e^{Ht}=-\frac{1}{H\tau}$$

$$\boxed{\begin{aligned}&\frac{\frac{da}{dt}}{a}=\frac{\frac{da}{a\,d\tau}}{a}=H=\text{const}\\&\frac{1}{a^2}\cdot da=H\,d\tau\\&-\frac{1}{a}=H\tau\\&a=-\frac{1}{H\tau}\end{aligned}}$$

推导:

Friedmann Eqn: ① $H^2 = \frac{1}{3M_{pl}^2}\rho = \frac{1}{3M_{pl}^2}(2X P_{,X} - P)$

$$2H\dot H = \frac{1}{3M^2}\left[2\dot X P_{,X} + 2XP_{,XX}\dot X + 2XP_{,X\phi}\dot\phi - P_{,X}\dot X - P_{,\phi}\dot\phi\right]$$

$$X = \frac{1}{2}\dot\phi^2$$

$$\therefore \dot X = \dot\phi\,\ddot\phi$$

② $\dot H = -\frac{XP_{,X}}{M^2}$

$$\therefore\ -3\frac{\dot\phi^2 P_{,X}}{M^2}H = \frac{1}{3M^2}\left[(~~\dot X~~P_{,X} + 2XP_{,XX})~~\dot\phi~~\ddot\phi + 2XP_{,X\phi}~~\dot\phi~~ - ~~P_{,X}\dot\phi\ddot\phi~~ - P_{,\phi}~~\dot\phi~~\right]$$

$$(2XP_{,XX} + P_{,X})\ddot\phi + 3H\dot\phi P_{,X} + 2XP_{,X\phi} - P_{,\phi} = 0$$

- Slow roll para: $\epsilon = -\frac{\dot H}{H^2} = \frac{3XP_{,X}}{2XP_{,X} - P}$

CH2.

$g_{\mu\nu}(t,\vec x) = \bar g_{\mu\nu}(t) + \hat h_{\mu\nu}(t,\vec x)$ $\qquad\phi(t,\vec x) = \bar\phi(t) + \hat\varphi(t,\vec x)$

场论扰动才是作为算符.

$g_{\mu\nu} = \begin{pmatrix} -1 & \\ & a^2\delta_{ij}^{(3)}\end{pmatrix}$

先考虑φ ○ Action: $S = -\int\sqrt{-g}\,\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi = \int d^3x\,dt\ a^3\cdot\frac{1}{2}\left[\dot\varphi^2 - \frac{1}{a^2}\partial_i\varphi\,\partial^i\varphi\right]$

这里已经是对 Comoving coordinate 求导.

$$S_k = \int\frac{d^3k}{(2\pi)^3}\,dt\ a^3\,\frac{1}{2}\left[\dot\varphi_{(k)}\dot\varphi_{(-k)} - \frac{k^2}{a^2}\varphi_{(k)}\varphi_{(-k)}\right]$$

○ Fourier Transform: $\varphi(x) = \int_k \varphi(\vec k)\,e^{i\vec k\cdot\vec x}$ $\qquad\varphi(\vec k) = \int_{\vec x}\varphi(\vec x)\,e^{-i\vec k\cdot\vec x}$

其中 $\int_k = \int\frac{d^3k}{(2\pi)^3}$ $\qquad\int_x = d^3x$

○ $\varphi(\vec k, t) = f_k(t)\,a_k + f_k^*(t)\,a_{-k}^\dagger$ [▫ Time Dependence 在 $f(t)$ 上]

Target: 导出 Mukhanov - Sasaki Eqn

具体量子化步骤 ref Baumann.

1. 得到 φ 的 EoM 2. 将其量子化 $\Longrightarrow \hat\varphi = f_k(t)\hat a_k + f_k^*(t)\hat a_{-k}^\dagger$ 要求 $f_k(t)$ 满足 EoM

$$\therefore\ \ddot f_k + 3H\dot f_k + \frac{k^2}{a^2}f_k = 0$$

$$(af_k)'' + \left(k^2 - \frac{a''}{a}\right)(af_k) = 0$$

P.S EoM: $\varphi(-k) \to \delta\varphi(-k)$ 就可以得到 EoM

由于运动空间不能直接用 E-L 方程.

Proof:

$dt = a\,d\tau$

$\dot f = \frac{df}{dt} = \frac{1}{a}f'$

$\ddot f = \left(\frac{1}{a}f'\right)^{\cdot} = -\frac{a'}{a^3}f' + \frac{1}{a^2}f''$

$\quad = -\frac{H}{a}f' + \frac{1}{a^2}f''$

$3H\dot f = \frac{3H}{a}f'$

原式 $= \frac{2H}{a}f' + \frac{1}{a^2}f'' + \frac{k^2}{a^2}f$

$(af)' = af' + a'f$

其中 $a' = \frac{da}{d\tau} = a\frac{da}{dt} = a\dot a = a^2H$

所以 $(af)'' = af'' + 2a'f' + a''f$

$\quad = af'' + 2a^2Hf' + a''f$

于是原式 $(af)'' + \left(k^2 - \frac{a''}{a}\right)(af) = 0$

$\because\ a = \frac{1}{H\tau}$

$\therefore\ \frac{a''}{a} = \frac{2}{\tau^2}$

$t\in(-\infty,\infty) \longleftrightarrow \tau\in(-\infty,0)$

- Baumann中 Superhorizon: $k \ll aH$
- $k\tau$ 来作为一个判据:

$aH = \frac{1}{|\tau|}$

$|\tau| = \frac{1}{aH} \sim$ Hubble radius

$k_{HC} := \frac{1}{|\tau|}$ 叫做 Hubble Crossing

☆ 当 Dim ≠ 3+1 时方程不是这种形式

$$a^2\underbrace{(n-1)}H f' + \underbrace{a}f'' + ak^2 f$$

系数不匹配了.

$(af_k)'' + (k^2 - \frac{2}{\tau^2})(af_k) = 0$

$\Rightarrow \quad f_k = \alpha(1+ik\tau)e^{-ik\tau} + \beta(1-ik\tau)e^{ik\tau}$

$k\tau >> 1$ 对应于很早期 — 早期宇宙.

- 这 EOM: $(af)'' + (k^2 - \frac{2}{\tau^2})(af) = 0$

  $\Rightarrow (af)'' + k^2(af) = 0$ 回到了K-G方程 [别从 f 的解中近似]

✻ Bunch Davies 真空: 早期宇宙 $a\varphi(x)$ 生活在 effective 的闵氏空间中.

定义 $\hat{\varphi}(x,t) = \int \frac{d^3k_p}{(2\pi)^3} \frac{e^{ik_p\cdot x}}{\sqrt{2k_p}} [e^{-ik_pt}\hat{a}_{k_p} + e^{ik_pt} a^\dagger_{-k_p}]$

- $k_p \equiv k/a$ physical wave number.
- 这个 $\hat{\varphi}$ 可产生正能量粒子: $\hat{H}\varphi(k)|0\rangle = ([H,\hat{\varphi}] + \hat{\varphi}\hat{H})|0\rangle$

Heisenberg Eqn: $-i\dot{\varphi}$ $\quad$ $\hat{H}|0\rangle = 0$ 我们选的真空.

$= k\hat{\varphi}(k)|0\rangle$

于是 f_k 要在 $k\tau \to 0$ 时得到该解.

$af_k \equiv \frac{e^{ik_p\cdot x}}{\sqrt{2k_p}}$ (当 $k\tau_* >> 1$)

同时还有高阶 $\partial_t(af_k) = \partial_t \frac{e^{ik_p\cdot x}}{\sqrt{2k_p}}$

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Claim: $\alpha = i e^{ik\tau_*(1+H\tau_*)} \frac{H}{\sqrt{2k^3}} [1 + \frac{i}{k\tau_*} - \frac{1}{2(k\tau_*)^2}]$ 习题第一

$\beta = i e^{-ik\tau_*(1-H\tau_*)} \frac{H}{\sqrt{2k^3}} \frac{1}{2(k\tau_*)^2}$

$\tau \to -\infty$ 时 $|\alpha| = \frac{H}{\sqrt{2k^3}}$ $\quad \beta = 0$

通过 近似EOM 得到确定
解系数的 B.C

于是 $f_k = \frac{H}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}$.

CH3 Weakly interacting field

- S矩阵: $S_{\alpha\beta} = \langle \alpha, \text{out} | \beta, \text{in} \rangle = \langle \alpha | S | \beta \rangle_H$ H代表 Heisenberg.
- State指标: 给定t后我们最关心的S 矩阵元. [illegible] 用守恒量本征值标记 α, β 等.

经典场论:

$S_{\alpha\beta} = 2^{\frac{n}{2}} \sqrt{E_1 E_2 \cdots E_n} \langle \Omega | a_{p_1 \text{out}} \cdots a_{p_n \text{out}} a^\dagger_{p_1 \text{in}} a^\dagger_{p_2 \text{in}} \cdots | \Omega \rangle$ 于是 $U_I(t, t_i) = T\{\exp[-i \int_{t_i}^{t} dt' H_{\text{int}}(t')]\}$

Prob ∝ $|\langle \alpha | S | \beta \rangle|^2$

相互作用表象: Def $|\psi, t\rangle_I = U_0(t, t_i) |\psi\rangle_S$ $S = U_I(\infty, -\infty)$

$O_I = U_0(t, t_i) O_S U_0^\dagger(t, t_i)$

·S与H $|\psi, t\rangle_H = U(t, t_i) |\psi\rangle_S$

$O_H = U(t, t_i) O_S U^\dagger(t, t_i)$

·形式上有I与H $|\psi, t\rangle_I = U_I |\psi\rangle_H$

$O_I = U_I(t, t_i) O_H U_I^\dagger(t, t_i)$

则 $U_I(t, t_i) = U_0(t, t_i) U^\dagger(t, t_i)$

$\frac{d}{dt_i} U$ $e^{i\int_{t_i}^{t} H_0 dt} \, e^{-i\int_{t_i}^{t} H dt}$

0

$U_0 (-iH_0) U^\dagger + U_0 U^\dagger iH$

宇宙学

不同于场论: ① Broken Poincaré Sym ② in-out ⇒ in-in formalism ③ Cosmic variance [只能观测到 Hubble volume内 / 宇宙的[illegible]

Correlators:

$\langle O \rangle = \langle \Omega | O | \Omega \rangle$

☆
1. O := equal-time product of operators 则不需要 Time Ordering
2. 结果为 Real，对应于 observable [场论中这个不是 observable
3. $|\Omega\rangle$ 是相互作用理论的真空，但 $\lim_{t \to -\infty} |\Omega\rangle = |0\rangle$

考虑绝热演化, $|\Omega\rangle$ 的[illegible]解得.

相互作用绘景: 导出 Feyn. Rules; 也称 Int 绘景.

$\langle\psi_a|\hat{O}|\psi_b\rangle$ [illegible] $e^{iHt}\hat{O}e^{-iHt}|\psi_b\rangle$

$\therefore |\psi_H\rangle = e^{iHt}|\psi_S\rangle$

- 薛定谔

[illegible]

薛定谔 [illegible] $|\psi_H\rangle = \hat{H}_S^{\,\prime}|\psi_H\rangle$

$O_H = e^{iHt} O_S e^{-iHt}$

$= \ldots = e^{iHt} O_I e^{-iHt}$

$O_I = e^{iH_0t} O_S e^{-iH_0t}$, $e^{iH_0t}e^{-iHt}O_H e^{iHt}e^{-iH_0t}$

$|\psi\rangle = e^{iH_0t}e^{-iHt}|\psi_S\rangle$

- 海森堡

[illegible]

依化算符: [illegible]

$U(t,t_0) = T\{\exp[-i\int_{t_0}^{t} H_I]\}$

$U^\dagger(t,t_0) = U(t_0,t)$

• 态 & 演化

关的 QFT. 认为 $|\Omega\rangle$ 在 $\tau \to -\infty$ 是 $|0\rangle$

∘ 遥远的过去是 Free Theory ∘ 现在演化到相互作用的真空 $|\Omega\rangle$

= 微观态在 Hilbert Space 上的演化;

[Heisenberg 只表了 $|\Omega\rangle$ 不变时; 而 $\hat{O}$ 是对——对 [illegible] 的意

∘ 将 $|\Omega\rangle$ 逆演到 $\tau \to -\infty$: $e^{-i\hat{H}_{int}(\tau-\tau_0)}|\Omega\rangle$

∘ 用 $\hat{H}_{int}$ 的本征态展开 $|\Omega\rangle$: $e^{-iE_0(\tau-\tau_0)}|0\rangle\langle 0|\Omega\rangle + \sum_{n\neq 0} e^{-iE_n(\tau-\tau_0)}|n\rangle\langle n|\Omega\rangle$

为了让 $|\Omega\rangle \to |0\rangle$ 则让 $\tau \to \tau(1-i\epsilon)$

E_n 和 $|n\rangle$ 是 $\hat{H}_{int}$ 本征值/矢.

于是 $U_I = T\{\exp[-i\int_{-\infty(1-i\epsilon)}^{\tau} d\tau'\,\hat{H}_{int}(\tau')]\}$

∘ $\langle O\rangle = \langle\Omega|U_I^\dagger(\tau,-\infty)\,O_I(\tau)\,U_I(\tau,-\infty)|\Omega\rangle$

Here it's In-In formalism

Recap Peskin:

态演化: $e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$.

$|n\rangle, E_n$ 是总能量 H 的本征值/矢.

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left[e^{-iE_0T}\langle\Omega|0\rangle\right]^{-1} e^{-iHT}|0\rangle$$

$$\lim_{T\to\infty(1-i\epsilon)} \left[e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right]^{-1} e^{-iH(T+t_0)}\,1\,|0\rangle$$

$$\because H_0|0\rangle = 0 \quad \therefore 1 = e^{-iH_0(-T-t_0)}$$

$$\therefore |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left[e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right]^{-1} e^{-iH(T+t_0)}\, e^{-iH_0(-T-t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left[e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right]^{-1} U(t_0,-T)\,|0\rangle$$

☆ $e^{-iHT}|0\rangle$ 是从无穷远过去演化到 [illegible]; 由于出发点是真空态 $|0\rangle$ $\therefore e^{-iHT}|0\rangle$ 并有相同 [illegible] 但这过程描述的是 入态

$$\therefore |\Omega\rangle_{in} = \lim_{T\to\infty(1-i\epsilon)} \left[e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right]^{-1} U(t_0,-T)\,|0\rangle$$

$$(|\Omega\rangle_{out})^\dagger = \lim_{T\to-\infty(1+i\epsilon)} \left[\left[e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right]^{-1} U(t_0,T)\,|0\rangle\right]^\dagger$$

$$\langle\Omega|_{out} = \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

Peskin书上 ${}_{out}\langle\Omega|\,\mathcal{O}\,|\Omega\rangle_{in} = \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 \cdot e^{-iE_0 2T}\right)^{-1} \cdot \langle 0|U(T,t_0)\,\mathcal{O}\,U(t_0,-T)|0\rangle$

In-In Formalism 设有 ${}_{out}\langle\Omega|$ 即是 $|\Omega_{in}\rangle^{+}$ ☆

$\therefore \langle \mathcal{O}\rangle \equiv {}_{out}\langle\Omega|\,\mathcal{O}\,|\Omega\rangle_{in} = \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2\right)^{-1} \cdot \langle 0|U^{+}(t_0,T)\,\mathcal{O}\,U(t_0,-T)|0\rangle$ (∵这里取了共轭 ∴无 $e^{-iE_0 2T}$)

令 $\hat{\mathcal{O}} = \mathbb{1}$, $|\langle 0|\Omega\rangle|^2 = 1$

$\therefore \langle \mathcal{O}\rangle = \langle 0|U^{+}(t_0,T)\,\mathcal{O}\,U(t_0,-T)|0\rangle$

P.S 对比 Peskin上

$$\langle\Omega|\,\mathcal{O}\,|\Omega\rangle = \lim_{T} \frac{\langle 0|U(T,t_0)\,\hat{\mathcal{O}}(t_0)\cdot U(t_0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

Peskin

关联函数

$1 = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \cdot \langle 0|\,U(T,-T)|0\rangle$

$\langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle =$

→ 海森堡表象下　　2 转换为相互作用表象

$$\lim_{T}\left[e^{-iE(T-t_0)}\langle 0|\Omega\rangle\right]^{-1} \cdot \langle 0|U(T,t_0)\,U(x^0,t_0)^{+}\,\phi_I(x)\,U(x^0,t_0)\cdot U(y^0,t_0)^{+}\,\phi_I(y)\,U(y^0,t_0)\cdot U(t_0,-T)|0\rangle\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}$$

$$\therefore \langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle = \lim \frac{\langle 0|U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

若引入时序算符将会简单

可把U合并

$$\langle\Omega|\,T\{\phi(x)\,\phi(y)\}\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\,\phi_I(y)\,\exp[-i\int_{-T}^{T}dt\,H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T}dt\,H_I(t)]\}|0\rangle}$$

微扰论形式

Claim: correlator还可写作 $\langle Q(t)\rangle = \sum_{N=0}^{\infty} i^N \int_{t_{in}}^{t} dt_1 \int_{t_{in}}^{t_1} \cdots \int_{t_{in}}^{t_{N-1}} \cdot \langle [H_I(t_N) \cdots [H_I(t_2), [H_I(t_1), Q_0(t)]] \cdots] \rangle$

$\langle \mathcal{O}(\tau)\rangle_{0^{th}} = \langle 0|\,\mathcal{O}(\tau)\,|0\rangle$

$\langle \mathcal{O}(\tau)\rangle_{1^{st}} = i\int_{-\infty}^{\tau} \langle 0|\,[H_{int}(\tau'), \mathcal{O}(\tau)]\,|0\rangle\, d\tau'$

§3.3 Example

令 $V = \mu\, \varphi(x)^3$

$$H_{int} = \int d^3x \sqrt{-g}\, \mu\, \varphi(x,\tau)^3$$

Fourier Transform

$$= a^4 \mu \int_{q_1 q_2 q_3} \varphi(q_1,\tau)\, \varphi(q_2,\tau)\, \varphi(q_3,\tau) \cdot (2\pi)^3 \cdot \delta_D^3(q_1+q_2+q_3)$$

其中 $\varphi(q,\tau) = f_q(\tau) \cdot a_q + f_q^*(\tau)\, a_{-q}^\dagger$

$f_q(\tau) = \frac{H}{\sqrt{2q^3}}(1+iq\tau)\, e^{-iq\tau}$

- Bispectrum / 3-pt correlator [考虑 φ^3 耦合的3线顶角]

$$\langle \varphi(k_1,\tau) \cdot \varphi(k_2,\tau)\, \varphi(k_3,\tau) \rangle = i \int_{-\infty}^{\tau} d\tau' \left\langle \left[H_{int}(\tau'),\, \varphi_{k_1} \varphi_{k_2} \varphi_{k_3}\big|_{\tau} \right] \right\rangle$$

* 对于 $O^\dagger = O$, $\langle [H_{int}, O] \rangle = 2i\, \mathrm{Im} \langle H_{int}\, O \rangle$

PF: $\langle H_{int}\, O \rangle - \underset{\langle O^\dagger H_{int}^\dagger \rangle}{\underset{\parallel}{\langle O H_{int} \rangle}} = \langle H_{int} O \rangle - \langle (H_{int} O)^\dagger \rangle = \langle H_I O \rangle - \langle H_I O \rangle^*$ QED

* 所有 equal time product 的 实空间 场算符是厄米的

$[\Phi_a(x,t),\ \Phi_b(y,t)] = 0$, $\varphi^\dagger = \varphi \Rightarrow [\varphi(x_1) \cdots \varphi(x_n)]^\dagger = \varphi(x_1) \cdots \varphi(x_n)$

* 考虑有Parity Sym的系统: $\vec{x} \leftrightarrow -\vec{x}$

$$(\varphi(k_1) \cdots \varphi(k_n))^\dagger = \left(\int_{x_1 \cdots x_n} e^{-ik_n x_n} \varphi(x_1) \cdots \varphi(x_n) \right)^\dagger = \varphi(k_1) \cdots \varphi(k_n)$$

$$= \int_{\cdots} e^{ik_n x_n}\, \varphi(x_1) \cdots \varphi(x_n)$$

$$= \varphi(-k_1) \cdots \varphi(-k_n)$$

- Correlator Fourier形式

$$-2\, \mathrm{Im} \int_{-\infty}^{\tau} d\tau'\, a^4(\tau') \int (2\pi)^3 \cdot \delta_D(q_1+q_2+q_3) \times \langle \varphi(q_1,\tau')\, \varphi(q_2,\tau')\, \varphi(q_3,\tau')\, \varphi(k_1,\tau)\, \varphi(k_2,\tau)\, \varphi(k_3,\tau) \rangle$$

- 关于Wick定理:

正规序.

记号: $\overset{\bullet}{\varphi}(q,\tau')\, \overset{\bullet}{\varphi}(k,\tau) = \varphi(q,\tau')\, \varphi(k,\tau) - :\varphi(q,\tau')\, \varphi(k,\tau):$

$$\therefore \boxed{\langle \overset{\bullet}{\varphi}(q,\tau')\, \overset{\bullet}{\varphi}(k,\tau) \rangle = \langle \varphi(q,\tau')\, \varphi(k,\tau) \rangle = f_q(\tau') \cdot f_k^*(\tau)\, (2\pi)^3 \delta_D^3(q+k)}$$

- Note

① 其中 $\hat\varphi(k,\tau) = f_k(\tau)\cdot\hat a_k + f_k^*(\tau)\cdot\hat a^\dagger_{-k}$

可知 假设 Normal Ordering 后 原式 $= [a, a^\dagger]$

Heisenberg Pic 下 升降算符关系一直保持，只有系数改变 → 这

结果为 $f_q(\tau')\cdot f_k^*(\tau)$

② 交换 (q,τ') 与 (k,τ) 的位置不能直接互换

于是 $\langle\varphi(q,\tau')\varphi(k,\tau)\rangle =$

$\langle\varphi(k,\tau)\varphi(q,\tau')\rangle^*$

③ 回顾 Wick Thm

$$\langle\prod_{a=1}^{2n}\varphi_a\rangle = \sum_{perms}\left[\langle\varphi_1\varphi_2\rangle\cdots\langle\varphi_{2n-1}\varphi_{2n}\rangle\right]$$

回到三点关联函数：

- $\langle\varphi(k_1,\tau)\varphi(k_2,\tau)\varphi(k_3,\tau)\rangle = i\int_{-\infty}^{\tau}d\tau'\langle[H_{int}(\tau'),\varphi(k_1,\tau)\varphi(k_2,\tau)\varphi(k_3,\tau)]\rangle$

$$= -2\,\mathrm{Im}\int_{-\infty}^{\tau}d\tau'\cdot\int_{q_1q_2q_3}a^4(\tau')\mu(2\pi)^3\delta_D^3(q_1+q_2+q_3)\langle\varphi(q_1,\tau')\varphi(q_2,\tau')\varphi(q_3,\tau')\varphi(k_1,\tau)\varphi(k_2,\tau)\varphi(k_3,\tau)\rangle$$

- 利用 Wick Thm; 仅需考虑连通图，即 $\overbrace{k\ q}$

$$= -2\mu(2\pi)^3\ 3!\ \mathrm{Im}\int_{-\infty}^{\tau}d\tau'\int a^4(\tau')\int_{q_1q_2q_3}\overbrace{q_1k_1}\ \overbrace{q_2k_2}\ \overbrace{q_3k_3}$$

- ∵ 有 Parity Sym ∴ $f_{-k}=f_k$ 又利用 $\delta(q+k)$ 和 $\int_q$ 后 $f_q^* = f_{-k}^* = f_k^*$

于是原式 =

$$-2\mu\cancel{(2\pi)^3}\ 3!\ \mathrm{Im}\left[\prod_{a=1}^{3}f^*_{k_a}(\tau)\cdot\int_{-\infty}^{\tau}a^4(\tau')\prod_{a=1}^{3}f_{k_a}(\tau')\right]$$

[illegible]

对称因子

P.S de Sitter Space:

$$ds^2 = \frac{-d\tau^2 + dx^i dx^j\delta_{ij}}{\tau^2H^2} = -dt^2 + e^{2Ht}dx^i dx^j\delta_{ij}$$

? $a(\tau) = e^{Ht} = -\frac{1}{H\tau}$ (de Sitter) 1.39 式 $a(t)\propto e^{Ht}$

$a(\tau)\propto(-\tau)^{-1}$ Baumann P28

$$= -2\mu(2\pi)^3\ 3!\ \mathrm{Im}\left[\prod^{3}f^*_{k_a}(\tau)\cdot\int_{-\infty}^{\tau}\frac{1}{(H\tau')^4}\prod_{a=1}^{3}f_{k_a}(\tau')\right]$$

- f_q 由 Mukhanov Sasaki Eqn 解得： $f_q(\tau) = \frac{H}{\sqrt{2q^3}}(1+iq\tau)e^{-iq\tau}$

原式 = $$\boxed{-\frac{3}{2}\frac{\mu H^2}{(k_1k_2k_3)^3}\mathrm{Im}\left\{\left[\prod_{a=1}^{3}(1-ik\tau)\right]\int_{-\infty}^{\tau}\frac{d\tau'}{\tau'^4}\prod_{a=1}^{3}(1+k_a\tau')\ e^{-ik_T(\tau'-\tau)}\right\}}$$

$k_T = k_1+k_2+k_3$

Claim. !

Note: ① 之前讨论虚时间时 $\tau \to -\infty(1+i\epsilon)$ 在这里导记 有指数压制 积分 trivial ∴ 该结果合理.

② 这里有一个结构：$\int_{-\infty}^{\tau}\frac{d\tau'}{\tau'^n}e^{-ik_T\tau'}$ [有 $\Pi(1+ik_a\tau')$ 乘在 $\frac{d\tau'}{\tau'^n}$ 上]

可通过分部积分由 $E_i(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}dt$ 生成

③ 给出结果. Euler常数及近似

$$\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\rangle = (2\pi)^3\delta_D(k_1+k_2+k_3)\frac{\mu H^2}{2(k_1k_2k_3)^3}\left[\sum_a k_a^3\left[\gamma_E-1+\ln(-k_T\tau)\right]+k_1k_2k_3-\sum_{a\neq b}k_a^2k_b\right]$$

Comment:

1. 关联函数的结构

$\langle\varphi(k_1)\cdots\varphi(k_n)\rangle = (2\pi^3)\,\delta_D^3\left(\sum k_a\right)\ \underline{B_n(k_1\cdots k_n)}$ 　结合. 通常记作 $\langle\varphi(k_1)\cdots\varphi(k_n)\rangle'$

2. B_3 与 k 的 norm 有关, 方向无关.

奇性：

3. $B_3 \sim k^{-6}$

4. 关联函数在 $k_1\cdots k_3$ 旋转下不变.

5. 关联函数 $\tau\to 0$ 时 对数发散

与经典对比

红外发散

(发散 $\ln\tau$.)

TimeOrder. ⋯

§ Quadratic Interaction:

给定 $H_{int} = \int_x a^4\frac{1}{4!\Lambda^4}(\partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi)^2$

∵ $\varphi = \int e^{-iq\tau}\varphi_q$

∴ $\partial_\tau\varphi \propto [-iq\varphi_q + \varphi_q']\exp$

↓

q 项. 消掉

CH4: $P(X,\phi)$ 理论

$\phi(x,t) = \bar{\phi}(t) + \varphi(x,t) \qquad \varphi \ll \bar{\phi}$

$\delta X = X - \bar{X} = \dot{\bar{\phi}}\dot{\varphi} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$

$L = P(X+\delta X, \bar{\phi}+\varphi)$

P.S. $S = \int\sqrt{-g}\left[\frac{1}{2}M_{pl}^2 R + P(X,\phi)\right]$

$X \sim \frac{1}{2}\left[\dot{\phi}^2 - (\partial_i\phi)^2\right]$

展开到 L_2：

$$L = P + P_{,\phi}\cdot\varphi + P_{,X}\left[\dot{\bar{\phi}}\dot{\varphi} - \underline{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi}\right] + \frac{1}{2}\left[P_{,XX}\delta X^2 + 2P_{,X\phi}\delta X\varphi + P_{,\phi\phi}\varphi^2\right]$$

L_1：背景场的EOM 即 $\delta L = 0$ 导出EOM

$$L_2: -\frac{1}{2}P_{,X}\,\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\left[P_{,XX}\dot{\bar{\phi}}^2\cdot\dot{\varphi}^2 + 2P_{,X\phi}\dot{\bar{\phi}}\underline{\dot{\varphi}\varphi} + P_{,\phi\phi}\varphi^2\right]$$

$$S_2: \int d^3x\,dt\cdot a^3\cdot\frac{1}{2}\left[(P_{,X} + P_{,XX}\bar{X})\dot{\varphi}^2 - P_{,X}\partial_i\varphi\partial^i\varphi - m^2\varphi^2\right]$$ 分部积分.

$$m^2 = 3HP_{,X\phi}\dot{\bar{\phi}} + \partial_t(P_{,X\phi}\dot{\bar{\phi}})$$

$$a^3\cdot\frac{1}{2}\left[(P_{,X} + P_{,XX}\bar{X})\dot{\varphi}^2 - P_{,X}\partial_i\varphi\partial^i\varphi - m^2\varphi^2\right]$$ ⟶ 类似massive scalar作用量

近似： Claim: $P_{,\phi}$ 都是有 slow-roll 参数抑制.

$$m^2 = 3HP_{,X\phi}\dot{\bar{\phi}} + \partial_t(P_{,X\phi}\dot{\bar{\phi}}) \simeq 0$$

例证：见 Lecture Note

- 于是 $S_2 \simeq \int d^3x\,dt\,a^3 \cdot \frac{1}{2} P_{,X} \left[1 + \frac{2P_{,XX}\bar{X}}{P_{,X}} \dot{\varphi}^2 - \partial_i\varphi\partial^i\varphi \right]$
- 正则 φ: $\varphi_c := \sqrt{P_{,X}}\,\varphi$ 　忽略X 取线性近似. 　$\bar{X} = \frac{1}{2}\dot{\bar\phi}(t)^2$ 　$\dot{\bar X} = \dot{\bar\phi}\ddot{\bar\phi}$

$$\dot\varphi_c = \sqrt{P_{,X}}\,\dot\varphi + \frac{1}{2}(P_{,X})^{-\frac{1}{2}}\varphi \cdot \cancel{(P_{,XX}\dot{X} + P_{,X\phi}\dot\phi)}$$

$\partial_i \bar X = 0$

$$\partial_i\varphi_c = \sqrt{P_{,X}}\,\partial_i\varphi + \frac{1}{2}(P_{,X})^{-\frac{1}{2}}\varphi \cdot \cancel{(P_{,XX}\partial_i X + P_{,X\phi}\partial_i\phi)}$$

且由于 $\dot{\bar\phi}$, $\ddot{\bar\phi}$ 较小.

于是 $S_2 \simeq \int d^3x\,dt\,a^3\, \frac{1}{2}\left[c_s^{-2}\cdot\dot\varphi_c^2 - \partial_i\varphi\partial^i\varphi\right]$ 　　$c_s^2 = \frac{P_{,X}}{P_{,X} + 2P_{,XX}\bar X}$

EOM: $\ddot\varphi + 3H\dot\varphi - \frac{c_s^2}{a^2}\partial_i\partial_i\varphi = 0$

- 当 $k \gg \frac{aH}{c_s}$ 时

$$\ddot\varphi - \frac{c_s^2}{a^2}\partial\partial\varphi = 0 \Rightarrow \varphi(x) \sim e^{\pm i c_s k_p t - i\vec{k}_p\cdot x}$$

色散关系: $\omega^2 = c_s^2 \cdot k_p^2$

⇒ c_s 描述 $\bar\phi$ 背景场上扰动 φ 的传播.

- Comment: 对比之前 bunch-davis 情况, 这里 $c_s \neq c = 1$; 所以在自然单位制下应用量纲分析把 c_s 表示.
- $f_k = \frac{H}{\sqrt{2c_s k^3}}(1 + i c_s k\tau)\, e^{-ic_sk\tau}.$

Frozen: 当 $c_s k\tau \sim 1$ 时 φ freeze out.

By baumann: $\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar\rho + \bar p)}$

演化方程: $\Phi'' + 3(1+w)\mathcal{H}\Phi' + wk^2\Phi$

$\dot\Phi$ = const / decay 解.

所以在 Super Horizon Scale 上 Freeze Out.

4.4. 例式

Einstein 场方程在 $ds^2 = a^2(\tau)\left[(1+2\Phi)d\tau^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right]$

下写出的方程 表形会导出的演变方程.

这里 $c_s k\tau \sim 1$ 即 Freeze Out 　[de Sitter Space 　$\tau \sim \frac{1}{aH}$

- Power Spectrum. 根据条件引前. 量纲分析得到 $P(k) = \frac{H^2}{2c_s k^3}.$

P.S Power Spectrum 定义: $\langle\varphi(k)\varphi(k')\rangle = (2\pi)^3\,\delta_D^3(k+k')\,P(k).$

- $P(X,\phi)$ 理论中三阶顶角:

$\phi(x,t) = \bar\phi(t) + \varphi(x,t)$

$\delta X = X - \bar X = \dot{\bar\phi}\dot\varphi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$ 　[注意到 δX 本身是 1阶 2阶混合!]

$\mathcal{L} = P(X+\delta X, \bar\phi+\varphi)$

$$L_3 = \frac{1}{6}\left(P_{,XXX}\cdot\delta X^3 + P_{,\phi\phi\phi}\varphi^3\right) + \frac{1}{2}\, P_{,XX\phi}\,\delta X^2\varphi + P_{,\phi\phi X}\,\varphi\,\delta X^2 + \frac{1}{2}P_{,XX}\cdot\delta X^2 + P_{,X\varphi}\,\delta X\varphi$$

$= \cdots\cdots$ [illegible]

Comment: · 由于保有 rotational sym ∴ 所以都要与 ∂^i 缩并.

- 由于 ∂_t 部分出来了, 不再有 Lorentz Sym. ⟶ $\bar{\varphi}(t)$的存在本身就破坏 Lorentz对称性.

原文: broke boost & time translation by chosing time-dependent vacuum.

- 略去所有 $P_{,\phi}$

$$L_3 \simeq \frac{1}{6}\, P_{,XXX}\,\dot{\bar{\varphi}}^3\dot{\varphi}^3 - \frac{1}{2}P_{,XX}\,\dot{\bar{\varphi}}\dot{\varphi}\,(\partial_\mu\varphi)^2$$

Bispectrum 计算?

CH5

EFT:

- 引力量子化会在一圈可重整 但依旧是非微扰方法.
- 用EFT描述引力非常合适.

引力 & EFT:

能标

引力适用范围: M_{pl}

Inflation能标: H , 由引力波限制: $H < 10^{-5} M_{pl}$

$E \ll \Lambda \ll E_0$

$\sim H$ 人为截断 $\sim M_{pl}$

- 将 $\phi \longrightarrow \phi_L + \phi_H$ （高/低频）

  $\omega > \Lambda$ 时 ϕ_L 消失. $\omega < \Lambda$ 时 ϕ_H 消失

- 路径积分: $\int D\phi_H D\phi_L \; e^{iS(\phi_L, \phi_H)} = \int D\phi_L \cdot \left[\int D\phi_H \, e^{iS(\phi_L, \phi_H)} \right] \equiv \int D\phi_L \, e^{iS_\Lambda(\phi_L)}$

  - $S_\Lambda(\phi_L)$: Wilsonian有效作用量.

  由于 ϕ_L 在 UV有截断 ∴ 由 $S_\Lambda(\phi_L)$ 计算得到的可观测量是 UV-finite的

- 目标:

  通过 S_Λ 定义一个 weakly couple theory. 可用近乎 free的理论描述.

☆ 对有效作用量进行算符展开:

$$S_\Lambda(\phi_L) = \int d^4x \cdot \sum g_a O_a$$

g_a: 耦合常数

O_a: 满足对称性的所有可能local operators.

O_a是 场和场导数乘积构成

- 进行量纲分析并在 Λ 附近消去无量纲常数:

$$[O_a] = \Delta_a, \quad [g_a] = 4 - \Delta_a$$

于是 g_a 写为 $\Lambda^{4-\Delta_a} \lambda_a$, λ_a 是 $O(1)$ 的.

- $\int d^4x\, g_a O_a = \int d^4x \frac{\lambda_a}{\Lambda^{\Delta_a-4}} O_a \sim \lambda_a \frac{E^{\Delta_a-4}}{\Lambda^{\Delta_a-4}}$

  而 O_a 则与考察能量 E 截断 Λ 有关.

  Eg: $S_n = \int d^4x \frac{\partial_\mu\phi\partial^\mu\phi}{2}$ 中 $[\phi]=1$ 即当能标为 E 时 $\phi^n \sim E^n$ $(\partial\phi)^{2n} \sim E^{4n}$.

- 于是在低能区: $E \ll \Lambda$, 对 $\Delta > 4$ 的算符称之为无关算符; $\Delta < 4$ 的称为有关;

  $\Delta = 4$ 称之为 Marginal Operator; 由圈图修正决定 $\Delta_a - 4 = 0^\pm$

Claim: 若理论中有一个 irrelevant operator, 则在经过圈图修正后会自然得到一系列其他的无关算符. 引入了无关算符的理论是不可重整的

记 P42

ADM formalism.

Target: GW仅有2个 DoF, 本来有10个 DoF, 4个Gauge的 还有4个 constraint eqn.

给定度规.

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_iN^i & N_i \\ N_i & h_{ij} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix}$$

易证 $g^{\mu\nu}$ 是 $g_{\mu\nu}$ 的逆

$\det[g_{\mu\nu}] = \det[g^{\mu\nu}]^{-1}$

对分块阵: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \simeq \begin{bmatrix} A & B \\ & D - CA^{-1} \cdot B \end{bmatrix}$

$\therefore \det = \det[A] \cdot \det[D - CA^{-1} \cdot B]$

体元.

可知 $\sqrt{-g} = \sqrt{h} \cdot N$

⁽³⁾R推R

- 在这种表示下定义类时 4-vec: $n_\mu = (-N, 0, 0, 0) \Rightarrow n^\mu n_\mu = -1$.

Claim:

extrinsic curvature: 经过一个矢量在超平面Σ上移动时空间部分改变: $K_{ij} \equiv n_{i;j}$

$$K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - {}^{(3)}\nabla_{(i}N_{j)}\right]$$

Claim: 用 Gauss-Codazzi Eqn 从 $^{(3)}R$ 得到 R:

$$R = {}^{(3)}R + \left(K_{ij}K^{ij} - K^2\right) - 2\nabla_\alpha\left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right)$$

ADM整作用量:

引力 $S = \frac{M_{pl}^2}{2} \int d^4x \sqrt{h} N \left[{}^{(3)}R + K_{ij}K^{ij} - K^2\right]$ 多做分部积分推

物质 $S = \int d^3x\, dt\, N\sqrt{h}\, P(X, \phi)$

Claim: N. N^i 无时间导数项, h_{ij} 获得一阶导.

ADM形式？

Constraint Eqn: $\begin{cases} \frac{\delta S}{\delta N} = 0 \\ \frac{\delta S}{\delta N_i} = 0 \end{cases}$ 这四个约束进一步限制DoF

SVT 分解

FLRW 度规: Homogeneous + Isotropic

最好用 ISO(3) isometry gp 描述.

§ 6.2 Symmetry:

对称性:

- $[Q,H]=0$ 如图:

$$\begin{tikzcd}
\text{Solution } A(t_1) \arrow[r, dash] \arrow[d, dash] & A(t_2) \arrow[d, dash] \\
\text{Solution } B(t_1) \arrow[r, dash] & B(t_2)
\end{tikzcd}$$

- Lagrangian描述对称性:

$\phi \to \phi + \Delta\phi$ 时 action不变 or Lagrangian变换了一个全微分 $\Delta\mathcal{L} = \partial_\mu F^\mu$

Recap Noether Current:

$$0 = \delta S = \int [\delta(d^4x)\mathcal{L} + d^4x\,\delta\mathcal{L}]$$

而 $\delta(d^4x) \simeq d^4x\,\partial_\mu \delta x^\mu$

$$\therefore \int d^4x \left[\partial_\mu \delta x^\mu \mathcal{L} + \frac{\delta\mathcal{L}}{\delta\phi}\cdot\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\cdot\delta(\partial_\mu\phi) + \delta x^\mu \cdot \partial_\mu\mathcal{L} \right]$$

$$= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi$$

$$= \int d^4x\,\partial_\mu(\cdots) + \int \underbrace{\left(-\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} + \frac{\delta\mathcal{L}}{\delta\phi}\right)}_{\text{Euler Lagrange Eqn}}\delta\phi + \partial_\mu \delta x^\mu \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L}$$

$$\therefore = \int \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\delta\phi + \delta x^\mu \mathcal{L}\right)$$

$$\Rightarrow \quad \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\ldots}\delta\phi + \delta x^\mu\mathcal{L}\right] = 0$$

于是 $j^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \delta x^\mu \mathcal{L}$

则 $J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\Delta\phi - F^\mu$?? 流不的流特选.

- 协变形式 J^μ: $\mathcal{J}^\mu := J^\mu(-g)^{\frac{1}{2}}$ 则 $\nabla_\mu \mathcal{J}^\mu = (\partial_\mu J^\mu)(-g)^{-\frac{1}{2}}$ GR Book: $A^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^\mu)_{,\mu}$

$$Q = \int \sqrt{h}\, J^\mu n_\mu\, d^3x$$

鲜于忆：

dS > Mink > AdS

4维放大对称空间难以排序.

Fluctuation:

$$S = \int d^4x \sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

$$\phi(t, \vec{x}) = \phi_0(t) + \varphi(t, \vec{x})$$

- ADM Formalism: $ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)\cdot(dx^j + N^j dt)$

  N: lapse function → 代表时间单位

  N^i: Shift Vector → 代表 下一个 Time Slice 和上一个 Time Slice 相对位移

  h_{ij}: 时空切片的3维空间的度规

- $$S = \frac{M_{pl}^2}{2}\int dt\, d^3x \sqrt{h}\, N \cdot$$ 缩记

$$\left[R^{(3)} + \frac{N^{-2}}{M_{pl}^2}\left(E_{ij}E^{ij} - E^2\right) - \frac{2}{M_{pl}^2}V - \frac{1}{M_{pl}^2}h^{ij}\partial_i\phi\partial_j\phi\right]$$

其中 $E_{ij} = \frac{1}{2}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i)$

$K_{ij} = N^{-1}\cdot E_{ij}$ 外曲率

- Fluctuation:

$$N = 1 + \alpha$$

$$N_i = \partial_i\psi + \beta_i$$

$$h_{ij} = a^2 e^{2\zeta}(\delta_{ij} + \gamma_{ij} + \partial_i k_j + \partial_j k_i + \partial_i\partial_j\lambda)$$

其中 $\partial^i\beta_i = \partial^i k_i = 0$, $\partial^i\gamma_{ij} = 0$ $\gamma^i{}_i = 0$

$$\phi = \phi_0 + \varphi$$

Claim: CMB上温度涨落来自Scalar Part: $\beta_i = k_i = 0$, $\gamma_{ij} = 0$

一共有5个标量 DoF: $(\psi, \alpha, \phi, \lambda, \zeta)$

↑ Scalar ϕ

- Gauge Transformation:

$$\delta g_{\mu\nu} = -\bar{g}_{\lambda\mu}\partial_\nu\epsilon^\lambda - \bar{g}_{\lambda\nu}\partial_\mu\epsilon^\lambda - \epsilon^\lambda\partial_\lambda\bar{g}_{\mu\nu}$$

$$\delta\phi = -\epsilon^\lambda\partial_\lambda\bar{\phi}$$

Gauge Transform: 重选坐标系 [在时空上重新编格子, 物理"量"不变

参考系变换: 换观测者. [换个观测位置 → 物理结果改变

ϵ^{λ}: 无穷小坐标变换　　作3+1分解: $\epsilon^0 = \chi$　　$\epsilon^i = \partial^i \xi$

Ex:

$$\begin{cases} \delta\alpha = -\dot{\chi} \\ \delta\psi = \chi - \dot{\xi} \\ \delta\zeta = -H\chi \\ \delta\lambda = -\frac{2}{a^2}\xi \\ \delta\varphi = -\dot{\phi}_0\chi \end{cases}$$

ADM formalism中5个标量的变换

一般让 $\lambda \longrightarrow 0$

$\varphi \longrightarrow 0$

[若让 $\zeta \to 0$ 叫 ζ Gauge]

由于 infinitesimal transformation 仅有2个 DoF

∴仅可消去2个标量.

Recap: $N = 1+\alpha$　　　　$N = 1+\alpha$　　Gauging

$N_i = \partial_i\psi$　　$\rightarrow$　　$N_i = \partial_i\psi$

$h_{ij} = a^2 e^{2\zeta}(\delta_{ij} + \partial_i\partial_j\lambda)$　　　$h_{ij} = a^2 e^{2\zeta}\delta_{ij}$

$\phi = \phi_0 + \varphi$　　　　ϕ_0

$N = 1+\alpha$　Lapse　描述的是时空3+1分解 "非物理"

$N_i = \partial_i\psi$　Shift　故而 α, ψ 叫做 Constraint variable

Ex: 代回作用量:　P.S. [illegible]仅用2阶

$$S_2 = M_{pl}^2 \int dt\, d^3x$$

$$\Big[-a^3(3-\epsilon)H^2\alpha^2 + (-2aH\nabla^2\psi - 2a\nabla^2\zeta + 6a^3H\dot{\zeta})\alpha + 2a(\nabla^2\psi)\dot{\zeta} - 3a^3\dot{\zeta}^2 - a(18a^2H\dot{\zeta} + \nabla^2\zeta)\zeta$$
$$- 9a^3H^2(3-\epsilon)\zeta^2\Big]$$

其中　$\epsilon = \dot{\phi}_0^2/2H^2$,　$\nabla^2 = \delta^{ij}\partial_i\partial_j$

∵ α, ψ 不含时间导数 $\Rightarrow$ Constraint Variable.　　何为 Constraint Variable?

Ex　$\frac{\delta S_2}{\delta\alpha} = 0$　$\frac{\delta S_2}{\delta\psi} = 0$ $\Rightarrow \begin{cases} \alpha = \frac{\dot{\zeta}}{H} \\ \nabla^2\psi = -H^{-1}\nabla^2\zeta + a^2\epsilon\dot{\zeta} \end{cases}$

求出了 α, ψ 再代入 S_2, 利用分部积分化简

$$S_2 = M_{pl}^2 \int dt\, d^3x \;\; \epsilon\, [a^3 \dot{\zeta}^2 - a(\partial_i \zeta)^2]$$

总结 S_2 这很 Simple ∵ 真正 DoF = 1.

Coord Transform. $\quad d\tau = \frac{dt}{a}$

Def: $\quad \zeta' \equiv d\zeta/d\tau$

$$z \equiv \sqrt{2\epsilon}\, M_{pl}\, a$$

$$S_2 = \frac{1}{2}\int d\tau\, d^3x\, z^2 [(\zeta')^2 - (\partial_i \zeta)^2]$$

- $u \equiv z\zeta$

$$S_2 = \frac{1}{2}\int d\tau\, d^3x\, [(u')^2 - (\partial_i u)^2 + \frac{z''}{z} u^2]$$ Ex

EOM:

$$u'' - \partial_i \partial^i u - \frac{z''}{z} = 0$$

- Fourier Transformation

$$u(\tau, x) = \int \frac{d^3 \vec{k}}{(2\pi)^3} [u_k(\tau)\, e^{i\vec{k}\cdot\vec{x}} + c.c.]$$

Ex $\quad u_k'' + (k^2 - \frac{z''}{z}) u_k = 0$

Slow Roll Condition? $\quad \epsilon \to 0 \Rightarrow H \simeq \text{Const}$ o

$$a = e^{Ht} = -\frac{1}{H\tau}$$

$$\frac{z''}{z} = \frac{2}{\tau^2}$$

于是 $u_k = \frac{1}{\sqrt{2k}} [C_1 (1 - \frac{i}{k\tau}) e^{-ik\tau} + C_2 (1 + \frac{i}{k\tau}) e^{ik\tau}]$

$$\zeta_k = \frac{H}{2 M_{pl} \sqrt{\epsilon k^3}} [C_1 (1 + ik\tau) e^{-ik\tau} + C_2 (1 - ik\tau) e^{ik\tau}]$$ Ex

- 在早期时候 $\tau \to -\infty$ $\quad$ [Early Time, 所有 mode 都是在 small distance 中的, 都在 Hubble Horizon 内]
  - mode ~ 平面波 $\quad$ [所有 mode 都在较小时间尺度不太受背景影响, $g_{\mu\nu} \longrightarrow \eta_{\mu\nu}$]
- 在晚期时候 $\tau \to 0$
  - $\zeta_{(0)} = \text{Const.}$

Early Time 仍真空涨落出的平面波

Late Time 一切模式都 Frozen.

Quantization:

$$u(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\left[u_k(\tau) e^{i\vec{k}\cdot\vec{x}}\,\hat{a}_k + h.c.\right]$$

条件： ① $[u(\tau,\vec{x}), u'(\tau,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$ P.S $u' = \frac{\partial L}{\partial u}$ 是正则动量.

$[x, p] = i\delta(x-y)$

Why 要求场变量正则对易?

② $[a_k, a_p^{+}] = (2\pi)^3\,\delta^3(\vec{k}-\vec{p})$

由量子化条件给出场变量限制：

Claim: 这个给出 wronskian det: $\boxed{u_k(\tau)u_k'^{*}(\tau) - u_k^{*}(\tau)u_k'(\tau) = i}$ 1 $\boxed{(u_k'^{*})^2 + \omega^2 (u_k^{*})^2 = 0}$ 2

③ 定义真空： 让 $\hat{a}_k|0\rangle = 0$ 即定义 $|0\rangle$, 但可能不止一种

$H = \frac{1}{2}\int d^3x\left[(u')^2 + (\partial_i u)^2 - \frac{z''}{z}u^2\right]$ —— 量子化的展开 ——> $H|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big[\{(u_k'^{*})^2 + \omega^2 (u_k^{*})^2\}\cdot a^{+}_{-k}a^{+}_{k}$
$+ (|u_k'|^2 + \omega^2 |u_k|^2)(2\pi)^3\delta^{(3)}(0)\Big]|0\rangle$

其中 $\omega^2 = k^2 - \frac{z''}{z}$; Part 2 是 Minkowski 时空零点能 ∴ 让 Part 1=1 以得到真空态

- 由 1, 2 知 $\tau \to -\infty$ 时 $\frac{z''}{z} \to 0$, $u_k' = -iku_k$;

$u_k \propto c_1\left(1-\frac{i}{k\tau}\right)e^{-ik\tau} + c_2\left(1+\frac{i}{k\tau}\right)e^{ik\tau}$

- u mode 定了, 可由它来定 ζ mode $u = z\zeta$

Claim: $\zeta = \frac{H}{2M_{pl}}\frac{1}{\sqrt{\epsilon k^3}}(1+ik\tau)\,e^{-ik\tau + i\theta}$ ↘ stochastic quantum phase. —> CMB的涨落.

* Power Spectrum:

- 在 late time 假设 ζ 是常数 ┌→ 归一化

可测量 · $\lim\limits_{\tau\to 0}\langle 0|\zeta_{\vec{k}}(\tau)\zeta_{\vec{q}}(\tau)|0\rangle = (2\pi)^3\cdot\delta^{(3)}(\vec{k}+\vec{q})\ \frac{2\pi^2}{k^3}\ P_\zeta(k)$ Power Spectrum 的"定义"

↳ Claim. 这是 SO(3) 保证的 与 Kolen 谱关系??

- 对 $\zeta \propto e^{i\theta}$

θ是 stochastic quantum phase

在期望值中可消掉.

P.S 关于 $\tau \to 0$ 描述 late time

$-\infty < t < +\infty$ physical

$\because dt = a\,d\tau$

$\therefore a = e^{Ht} = -\frac{1}{H\tau}$

∴ τ 渐近于 0 的过程描述 $t \to \infty$ 的过程.

- 计算 Power Spectrum:

$$P_\zeta(k) = \frac{H^2}{8\pi^2 \epsilon M_{pl}^2} = \frac{H^4}{(2\pi)^2 \dot{\phi}_0^2}$$

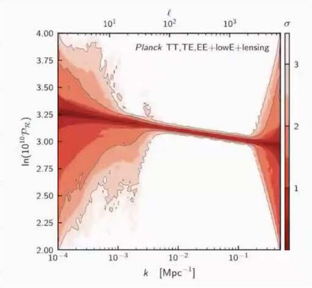
好!

- 可见 P independent of k ⟶ Scale invariant
- $P_\zeta(k)$ 是可观测量

  $P \sim A^2$ A 为扰动振幅

  $P \sim 10^{-9}$, $A \sim 10^{-5}$ Match!

  这里可见 Power Spectrum 微倾
  - 量级正确
  - 几乎跟 k 是 inv 的 [横轴是 log 坐标]
  - Claim: 斜率描述 Slow Roll



QFT in dS Background

∵ dS 中 H = const. • 这里将 H=1

$$ds^2 = \frac{-d\tau^2 + d\vec{x}^2}{\tau^2} \quad (H=1)$$

P.S. inflation model 有成百上千个
而观测量仅有几个
现在研究 model independent.

- Isometry GP of de Sitter. 什么是 Iso 群?

  3d Translation: $P_i = \partial_i$

  3d rotation: $J_i = \frac{1}{2}\epsilon_{ijk}(x_j\partial_k - x_k\partial_j)$

  Dilation: 对 the 4D space 同乘一个 scale factor

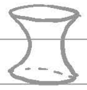
  $$D = -\tau\partial_\tau - x^i\partial_i$$

由于 dS 是最大对称空间 ∴ 一共有 10 个 Sym

- 对不同 generator commutator 是否能得到新的 Generator? No, 各个 Generator 都 close.

dS 想像成五维时空里的 hypersurface. Global dS

$$\boxed{\eta_{MN}X^M X^N = 1} \qquad M, N = 0, 1, \cdots 4$$



$$X^0 = -\frac{1-\tau^2+\vec{x}^2}{2\tau}$$

$$X^i = -\frac{x^i}{\tau} \qquad i = 1, 2, 3$$

$$X^4 = -\frac{1+\tau^2-\vec{x}^2}{2\tau}$$

SO(4,1) 的 Killing Vec:

$$J_{MN} = X_M \partial_N - X_N \partial_M$$ ← Global dS 的 Killing Vec

$$J_i = \frac{1}{2} \epsilon_{ijk} J^{jk}$$ ⇐ 与 Inflation Coord 对比

$$P_i = J_{i0} - J_{i4}$$ 对比小大

$$D = J_{40}$$

手迹 $K_i \equiv J_{i0} + J_{i4}$

Claim: 双曲面上的 const τ:

用 Global dS 做坐标变换 得到 Penrose Diagram

$$X^0 = \sinh\tau$$

$$X^i = w^i \cosh\tau \quad (i=1,4)$$

w^i: $w^1 = \cos\theta_1$

$$w^2 = \sin\theta_1 \cos\theta_2$$

$$w^3 = \sin\theta_1 \sin\theta_2 \cos\theta_3$$

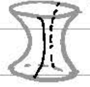
五维闵氏的这曲面满足一个方程

$$\Rightarrow \quad ds^2 = -d\tau^2 + \cosh^2\tau \, d\Omega_3$$

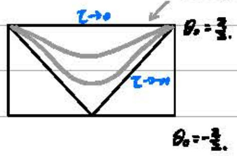
$$\cosh\tau = \frac{1}{\cos\theta_0} \qquad -\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$$

则 $ds^2 = \frac{1}{\cos^2\theta_0}\left(-d\theta_0^2 + d\Omega_3^2\right)$

Cut + 形变

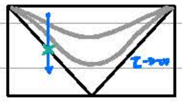


Poincaré Patch & 等τ线



每个点代表一个 sphere.

○ 是否解决 Singularity Problem?



Inflation Patch is Geodesically incomplete.

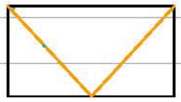
- Ref: 0110012
- 粒子测地线会从 inflation patch 中出去.

○ Inflation Patch Breaks full dS.

claim: 它破坏了 K_i

- 直观上它破坏了 z 轴的旋转对称性
- 可破坏了 $K_i = J_{i0} + J_{iq}$

Claim:



- 可以选一个 $|0\rangle$ 设在 $\tau \to -\infty$ 上
- 就走了 null surface

  $|0\rangle \longrightarrow |BD\rangle$
- 新真空态 respect 所有 Isometry
  - s.t. 从 $|0\rangle$ 开始演化到 BD 时 感觉不到对称性被破坏

QF & States in dS

找 State 的3种方法 ① 正则量子化 ② Path Integral ③ Weinberg + Wigner.

相互可以补充.

Case Study: Massive Scalar Field σ

$$(\Box - m^2)\,\sigma = 0$$

(bulk mode 多么重要)

$$\Rightarrow \sigma''(\tau,\vec{x}) - \frac{2}{\tau}\sigma'(\tau,\vec{x}) - \partial_i^2\sigma(\tau,\vec{x}) + \frac{m^2}{H^2\tau^2}\sigma(\tau,\vec{x}) = 0$$

$$\to \sigma_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\nu\pi/2} H(-\tau)^{3/2} \times H^{(1)}_{\nu}(-k\tau)$$

Hankel Function

$$\nu \equiv \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}$$

讨论 $m > \frac{3}{2}H$

$m < \frac{3}{2}H$

Early Time Limit: $\sigma_k(\tau) \sim e^{-ik\tau}$

Late Time Limit: $k\tau \to 0$ 对 Hankel 函数展开

$$\sigma_k(\tau) = -i\sqrt{\frac{2}{\pi k^3}}\, H \times \left[e^{-i\nu\pi/2}\,\Gamma(-\nu)\left(-\frac{k\tau}{2}\right)^{\frac{3}{2}+\nu} + e^{i\nu\pi/2}\,\Gamma(\nu)\left(-\frac{k\tau}{2}\right)^{\frac{3}{2}-\nu}\right]$$

$m \gg H$ 时，[ν是实的，会带 iν，ν 以 [illegible]]

带回t表象

$$\sigma_k(t) = \frac{1}{\sqrt{2\tilde{\nu} H}} e^{-3Ht/2} \times \left(e^{-i\tilde{\nu}Ht} b_k + e^{i\tilde{\nu}Ht} b_k^{\dagger} \right)$$

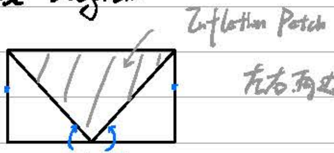
★ 在 massless 时 仅有 $(1-\frac{i}{k\tau})\cdot e^{-ik\tau}$

现在 massive 后 有了 $e^{i\tilde{\nu}Ht}$ Particle Production

代表 late time 粒子数从 0 变成非 0！

Class 3：

Penrose Diagram

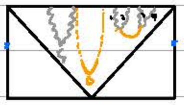


左右两边等同.

· 双曲面坐标系的 Penrose Diagram

Topol 与双曲面一致

在 τ→-∞ 时挑一个真空：$|0\rangle = |BD\rangle$



ζ: massless σ: massive

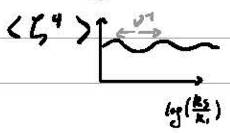
最不平凡的是 massive couple to massless

由于动量守恒

∴ 

用 $\langle\zeta^4\rangle$ 描述

$k_5 \ll k_1, k_2$



$\nu = \sqrt{\frac{9}{4} - (\frac{m}{H})^2}$ 当 $m > \frac{3}{2}H$ 时振荡

Back On Track:

σ obey K-G eqn. ，类似 u 的方程找到 σ_k 的解.

$\tau \to -\infty$ 时 $\sigma_k(\tau) \sim e^{-ik\tau}$

[illegible] 啥玩意？

$\tau \to 0$ 时 用 t 写出.

$$\sigma_k(t) = \frac{1}{\sqrt{2\nu H}} e^{-\frac{3Ht}{2}} \left(e^{-i\nu Ht} b_k + e^{i\nu Ht} b_k^\dagger \right)$$

这里 $b, b^\dagger$ 描述 t 的

$a, a^\dagger$ 描述 τ 的

↓ 长出了负频解.

- 粒子数并不守恒:

$n_k = \langle 0 | b_k^\dagger b_k | 0 \rangle$

Claim $\hat{b}_k$ 可写为 $\hat{a}_k$ 和 $\hat{a}_k^\dagger$ 展开

$$\hat{b}_k = \alpha_k \cdot \hat{a}_k + \beta_k \cdot \hat{a}_k^\dagger$$

则 $n_k = |\beta_k|^2$

Claim: $|\beta_k|^2 = \frac{\nu^2}{2\pi} |\Gamma(-i\nu)|^2 e^{-\nu\pi}$

考虑 $m \gg H$ 则 $\nu \simeq \frac{m}{H} \gg 1$

再对 Γ 级展开

$$|\beta_k|^2 = \frac{\coth \nu\pi - 1}{2} \simeq e^{-2\pi m/H}$$

Inflation 中产生粒子总是被 H 抑制. Inflation: 粒子制造机.

→ Cosmic Inflation can be used as an engine for particle production

可产生 $m \sim H$ 量级的粒子

Comment:

① $e^{-2\pi m/H}$: $|BD\rangle$ of dS is a thermal state

$e^{-m/T}$ where $T = \frac{H}{2\pi}$

② $\lim_{\tau \to 0} \sigma(\tau, \vec{x}) = \sigma_+(\vec{x}) (-\tau)^{\Delta_+} + \sigma_-(\vec{x}) (-\tau)^{\Delta_-}$

For scalar field, $\Delta_\pm = \frac{3}{2} \pm \nu$, $\nu = \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}$ ○

↳ dilute 指数速度

- 将 Isometry 作用在 $O(\vec{x})$ 上

$$P_i O = \partial_i O$$
$$J_{ij} O = \frac{1}{2}\epsilon_{ijk}(x_j\partial_k - x_k\partial_j) O(x)$$
$$D O = (-\Delta - x^i\partial_i) O$$
$$K_i O = (-2\Delta x_i - 2x_i x^j\partial_j + \vec{x}^2\partial_i) O$$

Claim: dS Isometry ≅ Future infinity 的 3d slice 上的 conformal gp 一一对应.

dS / CFT ? [不一定 Boundary 和 Bulk 的 Sym 相同，但不一定存在一个 Dynamically realization]

- Claim: UIR

Classification of state in dS 对应 Unitary Irreducible rep of SO(4,1)

SO(4,1)

SO(3) SO(1,1)

s: spin Δ: conformal weight

分类:

$s=0$

Principal series $m \geq \frac{3}{2}H$ $\Delta = \sqrt{\frac{9}{4} - (\frac{m}{H})^2}$

Complementary series $0 < m < \frac{3}{2}H$

Discrete Series 标量无 discrete series

$s \neq 0$

p $(\frac{m}{H})^2 > (s-\frac{1}{2})^2$ $\Delta = \sqrt{(s-\frac{1}{2})^2 - (\frac{m}{H})^2}$

c $s(s-1) < (\frac{m}{H})^2 < (s-\frac{1}{2})^2$

d

↓

Higuchi Bound

- 若写一个 high spin Lagrangian 让 mass 在下限下
  当违反 bound 时会发现 mode function 有 ghost.
- 但在 discrete series 中 $\begin{cases} (\frac{m}{H})^2 = s(s-1) - t(t-1) \\ t = 0, \cdots, s-1 \end{cases}$

- For $s=1$, or 2 massless mode是 discrete mode
  ↓ Photon ↓ Graviton

For $s=0$, $m=0$ is not in 1 particle state of dS

"Allen": there exists no minimally coupled massless scalar field 1 particle state in dS

- ζ mode late time limit 会趋于常数 → 导致CMB扰动
- ζ mode 逃出视界后会变成经典场

Claim: 无质量标量场 逃出视界 冻结的过程 会 break dS 的对称性

而 Goldstone, Axion 等在低能理论下都是 massless 的，要怎么解释？

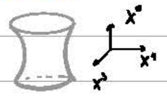
Remark: ① inflaton / ζ mode not affected because slow-roll 已经破坏了 dS 的 isometry

② Derivatively coupled massless scalar field 也不被上面这个影响.

而 Goldstone / Axion 都是导数形式耦合.

可以理解，对常数场一平移就 OK 了.

Embedding Distance:



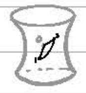
- $Z = g^{MN} X_M X_N$ 叫 embedding distance
- 在 inflation Coordinate 下

$$Z_{12} = \frac{\tau_1^2 + \tau_2^2 - |x_1 - x_2|^2}{2\tau_1\tau_2}$$

$Z < 1$ spacelike

$Z = 1$ null

$Z > 1$ timelike



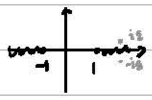
Geodesic distance $L_{12} = \arccos Z_{12}$

O1、O2 认为是单位长度

Z_{12} 为 $\vec{O1}$ $\vec{O2}$ 内积

Question: 由于 Z_{12} 可大于1 导致 $\arccos Z_{12}$ 无定义

⇒ 不能算两点连接

arccos z  arccos z 的 branch.

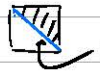
- 选择一种 'Prescription'

  对于 timelike 的 Z_{12}，要引入 Time-Order

  $\tilde{Z}(x_1,x_2) = Z(x_1,x_2) + \mathrm{sgn}(t_1-t_2)\times i\epsilon$ ——> 当在割线上 or 下 决定 $\pm i\epsilon$

  P.S. 后续 Path Integral 时也要引个ε，与这里ε一致

- $|BD\rangle$: Vacuum state BD in dS is dS-inv的

 $|BD\rangle$ $\hat{a}|BD\rangle=0$

引入 $|BD\rangle$ 时: ① 在 null surface上引入一个 $|BD\rangle$ state

② 选 conformal time 坐标并在此定义 $\hat{a}$ $\hat{a}^\dagger$

③ 让 $a|BD\rangle=0$

- 由于选的 inflation patch 等都破坏了 dS对称性

$|BD\rangle$是否是 dS-inv的?

证明 ① 用 Isometry Operator作用验证

② $\langle BD|\phi(x_1)\,\phi(x_2)|BD\rangle \equiv G(x_1,x_2)$

若 $G(x_1,x_2)$ 仅依赖于 $Z(x_1,x_2)$ 则可验证 $|BD\rangle$是 dS inv

↓ Geodesic distance

$$G(x_1,x_2) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} \times \langle 0|\phi_k(\tau_1)\,\phi_{-k}(\tau_2)|0\rangle$$

$$= \int \frac{k^2}{(2\pi)^2}\,dk \int d\cos\theta\; e^{ikx_{12}\cos\theta}\; \phi_k(\tau_1)\,\phi_k^*(\tau_2)$$

$$= \frac{e^{-\mathrm{Im}\,\nu\pi}}{8\pi x_{12}}\, H^2(\tau_1\tau_2)^{\frac{3}{2}} \int dk\, k\, \sin(kx_{12})\, H_\nu^{(1)}(-k\tau_1)\, H_{\nu^*}^{(2)}(-k\tau_2)$$ 难解析

当 $\nu=\frac{1}{2}$ 时

$$G(x_1,x_2) = \frac{H^2\tau_1\tau_2}{4\pi^2(x_{12}^2-(\tau_1-\tau_2)^2)} \quad \frac{H^2}{8\pi^2(1-Z_{12})}$$ 仅与 Z_{12} 有关 → dS-inv的

Claim: 对任意 ν, 对任意 D

$$G(x_1,x_2) = \frac{1}{(4\pi)^{D/2}}\, \frac{\Gamma(\mu_+)\Gamma(\mu_-)}{\Gamma(D/2)} \times {}_2F_1\left(\mu_+,\mu_-;\frac{D}{2};\frac{1+Z_{12}}{2}\right)$$

$$\mu_\pm = \frac{D-1}{2} \pm \sqrt{\frac{(D-1)^2}{4}-\left(\frac{m}{H}\right)^2}$$

Question: How to obtain $G(x_1,x_2)$ for general ν?

上面这 $\int dk\, k\, \sin(kx_2)\, H_\nu^{(1)}(-k\tau_1)\, H_{\nu^*}^{(2)}(-k\tau_2)$ 给到 $G(x_1,x_2)$ 要该怎么算？ 直接解 K-G方程

$$(\Box_{x_1}^2 - m^2)\, G(x_1,x_2)=0$$

假定 $G(x_1,x_2)=G(z)$

$$\Rightarrow (1-z^2)\,G''(z) + Dz\,G'(z) - \left(\frac{m}{H}\right)^2 G = 0$$

$$\Rightarrow G(z) = C_1\, {}_2F_1\left(\mu_+,\mu_-;\frac{D}{2};\frac{1+z}{2}\right) + C_2\, {}_2F_1\left(\mu_+,\mu_-;\frac{D}{2};\frac{1-z}{2}\right)$$

如何选 C_1 C_2 呢？

而 ${}_2F_1(a\,b\,c;\xi)$ 有一个 pole 在 $\xi=1$ 即 $z=\pm1$ 时的 pole.

Claim: C_1 C_2 任意组合都可在 early-time 定义真空态.

仅当 $C_2=0$ 时才是 BD 真空

XYZZ: there exist a family of dS-inv vacua, parametrized by $\alpha=\frac{C_2}{C_1}$

$\alpha=0 \to$ BD vacuum. ; α-vacua 都保留 dS symmetry，但并不都是物理的.

- dS 指数膨胀，里面粒子态被稀释. 里面的态依不再能是 dS-inv的
演化下的 |BD> 从这个角度看是 dS-inv的.
- α vacuum: 每一个能量都有粒子态，在被稀释时 还保持不变

|BD> vacuum 是 thermal 的

之前提到 $|\beta_k|^2 \simeq e^{-2\pi m/H}$　这里更详细的描述

Unruh detector

直觉：一个 observer 随身带一个 盖格计数器.

将盖格计数器制作 Hilbert Space 中一些态

在这个 Hilbert Space 上 引入相互作用

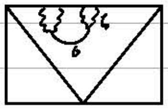
$$I = \int d\tau\, g\, m(\tau)\, \phi(x(\tau))$$

$|BD\rangle|E_i\rangle \longrightarrow |\beta\rangle|E_j\rangle$

σ observer $\quad\quad\quad\quad \downarrow \sigma$

. . .

Part 4:



通过测量 $\langle\zeta^{(n)}\rangle$反推 σ-mode 质量 or etc

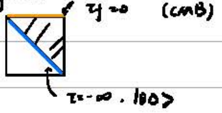
Cosmological Collider

- SM Higgs 125 GeV
  $H \lesssim 10^{14}$ GeV
- dS 中 不存在 minimally coupled massless scalar field [Zn CH 3]

Path Integral 1703.10166

目标： 计算 $\langle\varphi(\tau_f,\vec{x}_1)\cdots\varphi(\tau_f,\vec{x}_n)\rangle$

Penrose Diagram:



Target: 计算 $\langle BD|\varphi_1\cdots\varphi_n|BD\rangle \xrightarrow{\tau:-\infty\to0} \langle\varphi(\tau_f,\vec{x}_1)\cdots\varphi(\tau_f,\vec{x}_n)\rangle$

Idea: $\langle BD|\cdots|BD\rangle \longrightarrow \Sigma(S_{\text{matrix}})^2$

in-in formalism $\quad\quad$ in-out formalism

Skill：在 $\tau_f = 0$ 时插入 $\mathbb{1} = \sum_{\alpha} |\text{out}\rangle\langle\text{out}|$

于是 $\langle BD|\varphi_1 \cdots \varphi_n|BD\rangle$

$= \sum_{\text{out}} \langle BD|\text{out}\rangle \langle\text{out}|\varphi_1 \cdots \varphi_n|BD\rangle$

$= \sum S_{\text{matrix}}^{+} \cdot S_{\text{matrix}}$

而 $\langle\text{out}|\cdots|\text{in}\rangle = \int D\varphi \cdots e^{iS[\varphi]}$

于是 用2套场描述

$\langle BD|\varphi_1 \cdots \varphi_n|BD\rangle =$

✓ Claim. 上面 $\sum_{\text{out}}$ 自动进行了

$\int D\varphi_- D\varphi_+ \, e^{iS[\varphi_+]-iS[\varphi_-]} \times \prod_{\vec x} \delta[\varphi_+(\tau_f,\vec x) - \varphi_-(\tau_f,\vec x)] \, \varphi_{1+} \cdots \varphi_{n+}$ 对应 $\delta[\varphi_+ - \varphi_-]$

在 τ_f 的截面上让 φ_+ 与 φ_- 相同

其中 φ_- 描述 $\langle BD|\text{out}\rangle$

φ_+ 描述 $\langle\text{out}|\varphi_1 \cdots \varphi_n|BD\rangle$

推导 Feynman Rules

Propagator & Vertex

$$\left.\begin{aligned} G_{++}(x_1,x_2) &= \langle 0|T\{\varphi(\tau_1,\vec x_1)\varphi(\tau_2,\vec x_2)\}|0\rangle \\ G_{--} &= \langle 0|\bar T\{\cdots\}|0\rangle \\ G_{+-}(\cdots) &= \langle 0|\varphi(\tau_2,\vec x_2)\,\varphi(\tau_1,\vec x_1)|0\rangle \\ G_{-+} &= \langle 0|\varphi(\tau_1,\vec x_1)\,\varphi(\tau_2,\vec x_2)|0\rangle \end{aligned}\right\} \boxed{\text{Bulk Propagator}}$$

正场代到正行

动量空间：

- 注意，仅能在 3d 做 Fourier transform 得到 3d momentum space.

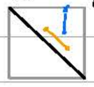
$G_{\underset{\pm}{\underline{ab}}}(k;\tau_1,\tau_2) = -i\int d^3\vec x \, e^{-i\vec k\cdot\vec x} G_{ab}(x_1,x_2)$

$G_<(k,\tau_1,\tau_2) = u_k(\tau_1)\,u_k^*(\tau_2)$ Claim

则 $G_{++} = G_> \theta(\tau_1-\tau_2) + G_< \theta(\tau_2-\tau_1)$ 其中 $G_> = G_<^{+}$

Bulk-to-Boundary Propagator

两个 Boundary φ_+ 和 φ_- [SK formalism 中有 $\delta[\varphi_+ - \varphi_-]$]



- bulk to boundary
- bulk

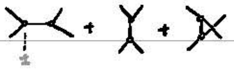
所以也就有 propagator 仅有 G_+ or G_- ⇒ $G_\pm(k,\tau)$ ∵ τ_f 固定

其中 $G_\pm = G_{\pm +}(k;\tau,\tau_f)$

Vertex

 $= i\lambda(\partial_\tau\varphi_1\,\partial_\tau\varphi_2\,\partial_\tau\varphi_3)$  $= -i\partial_\tau(---)$

Feynman Diagram

4: 4-pt Correlator $\sum_{\pm}$ 

Eg: 3-pt correlator of ζ-mode in slow-roll inflation. (Bispectrum)

- 先定义 Shape Function

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle' = \frac{(2\pi)^4 P_\zeta^2}{(k_1k_2k_3)^2}\cdot \underbrace{S(k_1,k_2,k_3)}_{\text{Shape Function}}$$

由动量守恒. $S(k,\cdots)$ 是 $S(\triangle)$

∵有δ函数让 k 守恒 ∴ k 能构成矢量三角形 

Claim: Shape function 依赖三角形的 Shape, 不依赖 k 大小 ⇒ $S(k_1,\cdots) = S(\frac{k_2}{k_1},\frac{k_3}{k_1})$

于是 S 是 Scale-Invariant 的

给定了的 S. 计算 Bispectrum

$$S = \int d^4x \sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

Claim:

$$S_{(3)} = M_{pl}^2 \int d\tau\, d^3x \left\{a^2\epsilon^2\left[\zeta(\zeta')^2 + \zeta(\partial_i\zeta)^2 - 2\zeta'\partial_i\zeta\underset{\sim}{(\partial_i^{-1}\zeta')}\right] - \frac{1}{2}\partial_\tau\left(\epsilon\eta a^2\zeta^2\zeta'\right)\right\}$$

还有 Higher Order ϵ, η

astro-ph 0210603

Why: Prob. 1523

E.g:

Interaction Term: $M_{pl}^2 a^2 \epsilon^2 \zeta(\zeta')^2$.

则 $\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle'$ ' 意味无 δ

动量空间 Propagator: 对所有圈动量积分, 在 Vertex 上放 δ 函数

位形空间 : 对所有 vertex 对应的坐标积分

Here: 这里有3个 k, 一个 t.

$$\therefore \langle\cdots\rangle = 2iM_{pl}^2\ \epsilon^2 \sum_{a=\pm} a \int_{-\infty}^{\tau_f}\frac{d\tau}{(H\tau)^2}\times\left\{G_a(k_1,\tau)\,\partial_\tau G_a(k_2,\tau)\,\partial_\tau G_a(k_3,\tau) + 2\text{perms}\right\}$$

$$= \cdots$$

$$= \frac{H^4}{4M_{pl}^2\epsilon k_1^3 k_2 k_3 k_t} + 2\text{perms}$$

则 $S_{(1)} = \epsilon\left(\frac{k_1k_2}{k_3k_t} + 2\text{perms}\right)$ 其中 $k_t = |k_1| + |k_2| + |k_3|$

$\downarrow$ 第一项

$$S = \epsilon\left(\frac{k_1k_2}{k_3k_t} + 2\text{perms}\right) + \frac{\epsilon}{8}\left(\frac{k_1}{k_2} + 5\text{perms}\right) + \frac{\eta-\epsilon}{8}\left(\frac{k_1^2}{k_2k_3} + 2\text{perms}\right)$$

Answer.

Remarks: ① 所有项都 slow-roll suppressed. $\epsilon, \eta < 10^{-2}$

② $\langle\zeta^3\rangle$ 由 Graviton Exchange 主导.

1811.00024

1210.7792

50 min class 4

直观上何为 ζ:

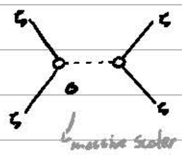
$$ds^2 = -dt^2 + e^{2Ht + 2\zeta(t,x)}dx^2$$

ζ 视作对 Hubble 的 Perturbation ⟶ 控制宇宙不同时间点膨胀速度不同 ⟶ CMB 扰动

4pt Function:

① 选定 Gauge. 令 $\zeta \to \varphi$ 中的扰动 一般知道 φ 与 σ couple 而非 ζ.

$\zeta \Rightarrow -\frac{H}{\dot\phi_0}\varphi.$



给定相互作用项: $\frac{1}{\Lambda}\sqrt{-g}(\partial_\mu\varphi)^2\sigma$ 引入导数为了让体系有 Shift Sym

从而让 Power Spectrum 有 Scale Invariance

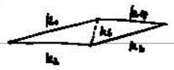
先考虑其中 $\frac{1}{\Lambda}a^2(\varphi')^2\sigma$ 项 故而称为 σ 与 Inflaton 的 Derivatively Coupled

$$\langle\varphi_{\vec k_1}\cdots\varphi_{\vec k_4}\rangle' = \frac{1}{\Lambda^2}\frac{1}{16k_1\cdots k_4}\sum_{a,b=\pm}\int_{-\infty}^{0}d\tau_1\int_{-\infty}^{0}d\tau_2\, e^{iak_{12}\tau_1+ibk_{34}\tau_2}D_{ab}(k_s;\tau_1,\tau_2)$$

$k_{12}=k_1+k_2$ 这是 σ 的传播子 可写成 mode function 求和

计算: ref 1811.00024 有解析计算; [由于 D 中还有 time ordering ∴ 难算]

Simple Case:



考虑 坍缩 Case: $k_s \ll k_1\cdots k_4$

$$\lim_{\tau_{1,2}\to 0}D(k,\tau_1,\tau_2) = D_{\text{local}} + D_{\text{non-local}}$$

Claim: Hankel Function 在 late time 会 Oscillate

$$D_{nl}(k;\tau_1,\tau_2) = \frac{H^2}{4\pi}(\tau_1\tau_2)^{\frac{3}{2}}\times\left[\Gamma^2(-i\tilde\nu)(k^2\tau_1\tau_2)^{+i\tilde\nu} + (\tilde\nu\to-\tilde\nu)\right]$$

考虑 $m > \frac{3}{2}H$，对应 $\tilde{\nu} = \sqrt{\left(\frac{m}{H}\right)^2 - \frac{9}{4}}$

则 $\langle \varphi_1 \cdots \varphi_4 \rangle'_{nl} = A(m,\Lambda) \frac{1}{k_1k_2k_3k_4(k_{12}k_{34})^{5/4}} \cdot \sin\left[\tilde{\nu} \log \frac{k_s^2}{k_{12}k_{34}} + \theta(\tilde{\nu})\right]$

Summary. 考虑 model 含 σ [massive scalar] 耦合；且 σ 的质量 $> \frac{3}{2}H$

则 $\langle \varphi_1 \cdots \varphi_4 \rangle' \propto \sin\left[\log \frac{k_s}{k_{12}k_{34}} + \cdots\right]$

Claim. 这正好代表 non-local 的信号

叫做 Cosmological Collider Signal

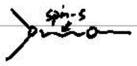
P.S 上面 $A(m,\Lambda) \underset{m\gg H}{\simeq} \frac{\pi H^2}{8\Lambda^2}\left(\frac{m}{H}\right)^3 e^{-\pi m/H}$ Boltzmann Suppression

Bispectrum:

i.e. $S(k_1,k_2,k_3) \sim \frac{\dot{\phi}_0}{\Lambda^2}\left(\frac{m}{H}\right)^{3/2} \cdot e^{-\pi m/H} \left(\frac{k_1}{k_3}\right)^{-\frac{1}{2}} \times \sin\left[\tilde{\nu}\log\frac{k_3}{k_1} + \varphi\right]$

Phase

Spin Particle

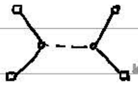


$\propto P_s(\cos\theta)$

Loop Effect

$\sim \sin\left[2\tilde{\nu}\log\left[\frac{k_s^2}{k_{12}k_{34}}\right] + \varphi\right]$

Notation:



代表 Boundary.

IR Problem:

Eg. $\lambda\phi^4$ 类的自相互作用

Claim: $\langle \phi^4 \rangle \sim \mathrm{Im} \int_{-\infty}^{\tau_f} d\tau\, a^4(\tau) \left[(1+ik\tau)e^{-ik\tau}\right]^4$

$\sim \int \frac{d\tau}{\tau^4}\cdot\tau^3$

$\sim \log\tau_f$

$\sim t_f$ 发散 [直观看，由于 mode function 最后趋于静止，所以 $\int d\tau$ 发散.

Wilsonian EFT in dS:

遇到的问题：若有一个截断 Λ，则随着宇宙学红移，mode 从 High E => Low E.

Bootstrap

SO(4,1) 的对称性中的 D 和 K 的对易关系来解 Amplitude.

但 Cosmology 中的 Bootstrap 对称性是被破缺。

dS: Horizon ~ BH Horizon 类似一个 Inside Out.

1703.10166

CH2:

度规: $ds^2 = a^2(\tau)(-d\tau^2 + d\vec{x}^2)$

背景场: 由 $\mathcal{L}_u[\phi]$ 描述。可能由多种场, EOM: $\frac{\delta \mathcal{L}}{\delta \phi} = 0$, $\phi^A = \bar{\phi}^A(\tau)$

其中我们需要每个 Σ_τ 上 $\phi =$ const. [即只依赖于τ]

场扰动 $\phi^A(\tau, \vec{x}) = \bar{\phi}^A(\tau) + \varphi^A(\tau, \vec{x})$.

Lagrangian 扰动 $\mathcal{L} \to \mathcal{L}_u[\varphi; \bar{\phi}(\tau)]$ 关于 φ^A 是 quadratic的.

记作 $\mathcal{L}[\varphi]$ for clarity, 认为其中所有 φ^A 都是 动力学变量

考虑 $\mathcal{L}_{cl}[\varphi]$ 没有高阶导数耦合项的 case

$$\mathcal{L}_{cl} = \frac{1}{2} U_{AB} \varphi'^A \varphi'^B + V_A(\varphi) \varphi'^A + W(\varphi)$$

其中 U_{AB} 是正定矩阵.

可得到 π_A ··· (7)

$H[\pi, \varphi]$ ··· (8)

Example:

$$\mathcal{L}_{cl}[\varphi] = \sum_a \left[\frac{1}{2} a^2(\tau) \varphi_a'^2(\tau, x) - \frac{1}{2} a^2(\tau) [\partial_i \varphi_a(\tau, x)]^2 - \frac{1}{2} a^4 M_a^2 \varphi_a^2\right] + \cdots$$

相互作用项.

量子化 得 $\hat{\varphi}_a(\tau, x) \propto \cdots [u_a(\tau, k) \hat{b}_a(\vec{k}) + u_a^*(\tau, -k) \hat{b}_a^\dagger(-k)]$

$$u_a(\tau, k) = -\frac{i\sqrt{\pi}}{2} e^{i\pi(\nu_a/2 + 1/4)} H \cdot (-\tau)^{3/2} H^{(1)}_{\nu_a}(-k\tau)$$

P.S 关于 Hankel Function:

汉克尔函数

第一类 $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ 渐近 claim 当 $x \to \infty$ $J \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\nu\pi}{2} - \frac{\pi}{4})$

第二类 $H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$ $Y \sim \sqrt{\frac{2}{\pi x}} \sin(x - \frac{\nu\pi}{2} - \frac{\pi}{4})$

$\therefore \lim_{x \to \infty} H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{[i(x - \frac{\nu\pi}{2} - \frac{\pi}{4})]}$ $\frac{1}{\sqrt{r}} e^{kr - \omega t}$ <u>柱面波</u>

$H_\nu^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{[-i(x - \frac{\nu\pi}{2} - \frac{\pi}{4})]}$

$\tau \to -\infty$ 时

$$\cancel{\sqrt{\frac{\pi}{2}}}\frac{1}{\sqrt{k\tau}}e^{[i(k\tau-\cancel{\frac{\nu\pi}{2}}-\cancel{\frac{\pi}{4}})]} \quad -\frac{i\cancel{\sqrt{2}}}{\cancel{\pi}\sqrt{2}}\, e^{i\pi(\cancel{\nu/2}+\cancel{1/4})} \quad H\cdot(-\tau)^{\frac{3}{2}} = \frac{iH\tau}{\sqrt{2k}}e^{-ik\tau} .$$

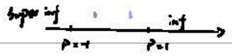
* 有质量粒子在极早期都会趋于massless极限.

SK path integral.

Cosmology Collider

Youtube

$a = t^p$



有很多 Inflation Model.

—: slow contraction

—: slow expansion

How To Test?

Model 太多了，于是考虑 model-indep 的研究.

1. 用GW来看 Inflation History.

① 不同 Inflation model 有不同 Energy scale

GW 的 Tilt 可指示 Energy scale for a certain development of time.

- 但这个依赖于一些 Assumption:

  1. Primordial GW assume 我们观测相当于测量 Const mode → EoM of GW 有2个，① const ② decay

     而模型 Matter Bounce 亦有相似预测

  2. Vacuum Fluctuation.

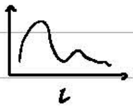
     我们认为 fluctuation 源自 vacuum, 但 string gas cosmology 不是这么认为

     ——源自 Holographic Thermal State.

     该结果也导致 const mode (近似 Scale-inv)

如何测量宇宙是 Expanding / Contracting.

先考虑更细的问题 如何处理观测量?



$k \sim -\frac{1}{\tau}$ 从 fluctuation 引出的总在 Conformal time

k 对应于 出/入 Horizon 的先后.

但 $k \sim k(t)$ 有难度. Physical Time —— Difficulty.

- 关于 $dt \sim a\,d\tau$: 这里 a 的演化 尚未完全确定.

Heavy Fields & Clock:

Massive Field 在 Potential 中振荡.

Mass 作为 Physical Para, associated with physical time.

所以 功率谱中提取振荡信号 可找到了 $t \sim \tau$ 的对应.

- math: $\int d\tau\, f(\tau)\, \underset{\text{Conformal}}{e^{-ik\tau}}\, \underset{\text{Physical}}{e^{imt}}$

对应的 $\Delta P_\zeta \sim \sin\Big[\underbrace{\cdots\left(\frac{k}{k_*}\right)^{\frac{1}{p}}}_{\text{找到这种信}} + \text{phase}\Big]$ Standard Clock

§ Alternatives to inflation as Cosmo Particle Scanner

§1

Particle Content of Early Universe

η - Problem $m^2 \sim \mathcal{O}(0.01)H^2$

May remain under H or higher

Not important Classically

But important quant---ly

(§3) Massive Particle Signal ?

Encode To Soft Limit ?

Bispectrum $S \xrightarrow{\text{Soft}} e^{-2\mu}\left(\frac{k_{\text{long}}}{k_{\text{short}}}\right)^{\frac{1}{2}\pm i\mu} P_0(\cos\theta).$

SM

SSB

BSM → High Spin

Neutrino

Gauge Boson

Primordial Standard Clocks

Inflation / Alternative

↳ · (incomplete)

- Correlation not unique
- Key Feature —— Tensor Perturb